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V.A. USPENSKY

PASCAL'S TRIANGLE

•

CERTAIN APPLICATIONS OF MECHANICS TO MATHEMATICS

Mir Publishers Moscow

This booklet, written by Prof. V. A. Uspensky, Dr. Sc. (Phys.-Math.). contains two lectures. The first lecture (*Pascal's Triangle*) is easy in understanding for the students of incomplete secondary (eight-year) schools, it discusses an important numerical table (called the Pascal's triangle) which is of use in solving a number of problems. Along with obtaining these solutions, the lecture touches upon the meaning carried by the expression 'to solve a problem'.

The second lecture (*Certain Applications of Mechanics to Mathematics*) is for the students of the 7th and 8th school forms. It deals with straightforward solutions of various mathematical problems (occasionally rather complex) by using certain laws of mechanics.

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ИЗДАТЕЛЬСТВО МИР

ПОПУЛЯРНЫЕ ЛЕКЦИИ ПО МАТЕМАТИКЕ

В. А. Успенский

ТРЕУГОЛЬНИК ПАСКАЛЯ

НЕКОТОРЫЕ ПРИЛОЖЕНИЯ МЕХАНИКИ К МАТЕМАТИКЕ

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V. A. Uspensky

PASCAL'S TRIANGLE
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CERTAIN APPLICATIONS OF MECHANICS TO MATHEMATICS

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PREFACE

A reader who is unfamiliar with what Pascal's triangle is should be given a warning that this is not at all a geometrical triangle with three angles and three sides. Pascal's triangle is a name for an important numerical table which is used to solve a number of numerical problems. Considering some of these problems, we shall touch upon what may be meant in general by the words 'to solve a problem'.

The given exposition does not assume any foreknowledge lying outside the scope of the incomplete secondary school curriculum, except for the knowledge of a definition and notation of the power with zero exponent. Namely, it is necessary to know that any number, not equal to zero, when raised to the zero power is assumed (by definition!) to be equal to unity: $a^0 = 1$ at $a \neq 0$.

* * *

The application of mathematics to physics (in particular to mechanics) is a matter of common knowledge—it suffices to open a school text-book. The higher sections of mechanics require complex and sophisticated mathematics.

There are, however, mathematical problems which can be successfully solved with the help of concepts and laws of physics. A number of problems of this kind solved by mechanical methods, i.e. problems involving the application of the laws of equilibrium, were presented by the author in his lecture "Solution of mathematical problems by mechanical methods" delivered at the Moscow State University in 1956 for students of the 7th and 8th school forms.

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PASCAL'S TRIANGLE

Translated from the Russian

by V. Kisin

Section 1. A problem suggested at the VIII Olympiad

The ninth- and tenth-grade participants of the VIII Moscow Olympiad (1945) were given the following problem[1]:

There is a network of roads (Fig. 1). 2^{1000} people depart from the point A. One half of them goes in an l direction, the other half in an m direction. On reaching the first street crossing each group is separated into two subgroups: one

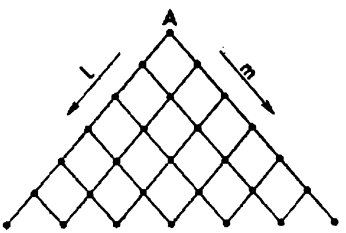


Fig. 1

half goes in the l direction, the other half in the m direction. The same division takes place at each crossing. How many people will arrive to each crossing of the 1000th row?[2])

Let us note first of all that we do not know yet whether the solution to this problem exists, i.e. whether the traffic of people can move in the way required by the conditions of the problem. If actually an odd number of people arrives at one of the crossings and the separation of the people

[1]) See Yaglom, A. M. and Yaglom, I. M. *Nonelementary Problems in Elementary Presentation.* Moscow, Gostekhizdat (1954). 'Numbers to problems suggested at Moscow Mathematics Olympiads', Problem 60b.

[2]) It should be borne in mind that the rows of crossings are numbered, beginning with the zero, so that there is only one crossing (A) in the zero row, two in the first row, three in the second, etc.

flow into two halves still is to take place, the traffic will have to stop. Hence, it is necessary and sufficient for the problem to have a solution that an even number of persons arrive at each street crossing of each of the first thousand rows, from the zero to the 999th. In solving the problem we shall demonstrate that this is exactly the case.

We begin with introducing the notations for the number of people passing through each street crossing in our network of roads. We shall enumerate the crossings of each row from left to right, beginning with the zero one; hence, the crossings of the nth row will have numbers from 0th to nth.

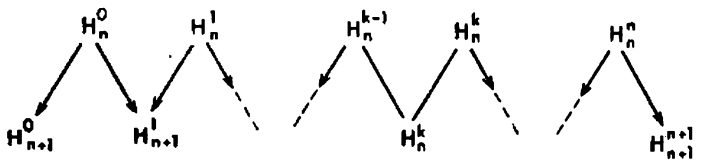


Fig. 2

The number of people passing through the kth crossing of the nth row we denote by H_n^k. Since we have not yet established whether the problem has a solution, we cannot be sure that *all* the numbers H_n^k exist, i.e. that each of the numbers H_n^k exists for any n from 0 to 1000 and for any k from 0 to n. At least some of them do exist. Thus by virtue of the introduced symbols, we have

$$H_0^0 = 2^{1000}. \tag{1.1}$$

Let us now consider how the numbers H_n^k ($k = 0, 1, 2, \ldots, n$) and H_{n+1}^k ($k = 0, 1, 2, \ldots, n+1$) are interrelated provided that all of them exist. We shall see that if all the numbers H_n^k exist and are even, then all the numbers H_{n+1}^k also exist. Let us consider now the nth and $(n+1)$th rows of cross-roads and the segments of roads connecting them; let us mark each crossing by the symbol denoting the corresponding number of people (see Fig. 2). The number of people departing from the 0th crossing of the nth row (i.e. H_n^0) will break evenly, one half coming to the 0th crossing of the $(n+1)$th row; therefore

$$H_{n+1}^0 = \frac{H_n^0}{2}. \tag{1.2}$$

The other half of H_n^0 will come to the first crossing of the $(n+1)$th row where it will join half the people that left the first crossing of the nth row, i.e. half of H_n^1. Therefore, $H_{n+1}^1 = \frac{H_n^0 + H_n^1}{2}$. And in general the number of people arriving at the kth crossing of the $(n+1)$th row is composed of half the number of people that departed from the $(k-1)$th crossing of the nth row (this half is equal to $H_n^{k-1}/2$) and half the number of people that departed from the kth crossing of the nth row (this half is equal to $H_n^k/2$). Thus

$$H_{n+1}^k = \frac{H_n^{k-1} + H_n^k}{2} \text{ for } 1 \leqslant k \leqslant n. \tag{1.3}$$

And, finally, the number of people arriving at the $(n+1)$th crossing of the $(n+1)$th row is equal to half the number of people that went out of the nth crossing of the nth row:

$$H_{n+1}^{n+1} = \frac{H_n^n}{2}. \tag{1.4}$$

The relationships (1.1)-(1.4) enable us to establish that the problem indeed has a solution. Actually, it follows from Eqs. (1.2)-(1.4) that if for any specified n all the numbers of the nth row: H_n^0, H_n^1, . . ., H_n^n, exist and are divisible by $2a$, then all the numbers of the $(n+1)$th row: H_{n+1}^0, H_{n+1}^1, . . ., H_{n+1}^{n+1}, exist and are divisible by a. Therefore, since all the numbers of the zero row (and there are only one of them—H_0^0) exist and are divisible by 2^{1000} (by virtue of Eq. (1.1)), all the numbers of the first row

$$H_1^0,\ H_1^1$$

exist and are divisible by 2^{999}; all the numbers of the second row

$$H_2^0,\ H_2^1,\ H_2^2$$

exist and are divisible by 2^{998}; . . .; all the numbers of the 999th row

$$H_{999}^0,\ H_{999}^1,\ \ldots,\ H_{999}^{999}$$

exist and are divisible by 2; all the numbers of the 1000th row

$$H_{1000}^0,\ H_{1000}^1,\ \ldots,\ H_{1000}^{1000}$$

exist (and are divisible by 1).

The relationships (1.2)-(1.4) not only prove the existence of a solution to the problem but also show how to obtain from a line of numbers

$$H_n^0,\ H_n^1,\ \ldots,\ H_n^n$$

a line

$$H_{n+1}^0,\ H_{n+1}^1,\ \ldots,\ H_{n+1}^{n+1}.$$

By successive application of these relationships, beginning with the zero line (i.e. using the relation (1.1)), we can, in principle, calculate the values of H_n^k for all 501 501 cross-roads present in the rows up to the 1000th one, inclusive, and for all the cross-roads of the 1000th row in particular, thereby bringing the problem to a solution. Thus, by straightforward calculations we obtain for the first several rows:

$$H_1^0 = \frac{H_0^0}{2} = \frac{2^{1000}}{2} = 2^{999};\quad H_1^1 = \frac{H_0^0}{2} = \frac{2^{1000}}{2} = 2^{999};$$

$$H_2^0 = \frac{H_1^0}{2} = \frac{2^{999}}{2} = 2^{998};\quad H_2^1 = \frac{H_1^0 + H_1^1}{2} = \frac{2^{999} + 2^{999}}{2} = 2^{999};$$

$$H_2^2 = \frac{H_1^1}{2} = \frac{2^{999}}{2} = 2^{998};\quad H_3^0 = \frac{H_2^0}{2} = \frac{2^{998}}{2} = 2^{997};$$

$$H_3^1 = \frac{H_2^0 + H_2^1}{2} = \frac{2^{998} + 2^{999}}{2} = 3 \times 2^{997};\ \text{etc.}$$

Section 2. What is meant by solving a problem?

Thus, the problem from Sec. 1 has been solved ...

'What do you mean, solved?' the surprised reaction of the inexperienced reader is (the experienced one knows in advance what the author will have to say, and is never surprised). 'I did not notice we had solved it.'

Author. Of course, we did. Solving a problem means finding its solution, doesn't it? And haven't we done just that?

Reader (*furiously*). Do you call this a solution?

Author (*pretending he does not understand what is the point*). Do you mean it is wrong?

Reader. No, 'it' is correct, only 'it' is not a solution at all.

Author. And what then is a solution?

Reader. A line of numbers showing how many people will arrive at the crossings in the thousandth row.

Author. But this line must contain 1001 numbers. The organizers of the VIII Olympiad could hardly hope that anyone of the participants would write out 1001 numbers, could they?

Reader (*deep in thought*).

Author. I have a suggestion to make. In order not to complicate the situation any further by long lines, let us select one of the crossings and let us concentrate on the number of people it received. Do you agree?

Reader (*agrees*).

Author. Now, what are we to accept as a solution to the following problem: how many people will pass through the third crossing in the fourth row?

Reader. How do you mean, what? A number.

Author. Written in what form?

Reader (*surprised*). Well, in decimal notation.

Author. And would you accept 'H_4^3' for a solution?

Reader. No, of course not. How can it be one?

Author. If we prolong the chain of arguments initiated at the end of the preceding Section, it will readily be seen that the number of people going through the third crossing of the 4th row is 2^{998}. Is '2^{998}' a solution of the problem?

Reader (*not realizing the trap yet*). Yes, certainly.

Author. But the expression '2^{998}' is not a decimal notation of the number. This expression consists of *two* decimal notations of numbers, viz. '2' and '998', and their relative locations indicate which operations must be performed over these numbers in order to obtain the required quantity.

Reader. However, the expression '2^{998}' can easily be transformed into a decimal notation.

Author (*didactically*). Not quite so easily. Just try to raise 2 to the 998th power. But this is rather beyond the point now; the thing is, you contradict here your own former statements. You yourself agreed to accept for a solution only a number presented in a decimal notation. From the standpoint of this definition the expression '2^{998}' is not yet a solution (it is, so to speak, a 'semifinished product' from which the solution should be manufactured). Such a viewpoint is, of course, possible provided it is applied in a consistent manner. However, another viewpoint is also possible according to which 2^{998} is a solution. This point of view should be more to your liking, since the answers to some mathematical problems are given fairly often not in a deci-

mal notation but in a similar to it 'indirect' form. But finally what interpretation of the term 'solution' should we choose for the problem concerning the third crossing of the 4th row?

Reader. In our case we must take as a solution any expression denoting a number and for which (expression, not number) there exists a method defining the transformation of this expression into a decimal notation of the corresponding number. 2^{998} is exactly such an expression. The method of transforming it into decimal notation (997 consecutive multiplications) is realizable in principle, although time-consuming.

Author. Then why H_4^3 is not a solution? It also contains the method of transformation to a decimal notation, given by Eqs. (1.1)-(1.4).

Reader (*he is confused*).

Author (*being glad that he managed to have his reader with his back to the wall. The reader is, of course, unsophisticated; a sophisticated one is capable of doing the same to the author*). The point is, there are, at least, three interpretations of what is the solution to the problem on the number of people that went through a given crossing:

FIRST INTERPRETATION. A solution is a number presented in a decimal notation.

SECOND INTERPRETATION. A solution is an expression denoting a number for which is available the method enabling us to transform this expression to a decimal notation of the number it represents (i.e. to the solution in the first interpretation).

THIRD INTERPRETATION. A solution is any such expression denoting a number which is composed of numbers (represented by a decimal notation) and of certain operations considered to be standard (for example, arithmetic operations)[1]). Let us require that each standard operation be accompanied by a method enabling us to transform a decimal notation of numbers to which it is applied into a decimal notation of the result of this operation (such are, for example, the arithmetic operations). Then there exists

[1]) The set of standard operations must be defined in advance. It is wise to emphasize that the third interpretation depends on the choice of this set. Thus, the expression 2^{998} will be the solution in the third interpretation as long as the involution is included into the set of standard operations.

:or the expression as a whole a method making it possible :o transform decimal notations of the numbers it involves :o a decimal notation of the number this expression signi- ies; thus a solution in the third interpretation will automa- ically be the solution in the second interpretation.

In the *first* interpretation neither H_4^3 nor 2^{998} is the solu- ion to the problem concerning the third crossing of the ourth row. To obtain the solution we have to find a decimal iotation for the number 2^{998}; however, the resulting figure nust contain more than 300 numbers, and the author has io information whether it was ever written[1]).

In the *second* interpretation both H_4^3 and 2^{998} are the solu- ions.

As for the *third* interpretation, in this case everything lepends on the choice of initial standard operations: if the nvolution is included into their number, 2^{998} will be a solu- ion, and will not if it is not included. In the same way, H_4^3 vill be the solution to the problem if the operation 'H' of alculating the number H_n^k for definite n and k is included nto the set of standard operations (the method of such a cal- ulation is given by the relationships (1.1)-(1.4); thus, the equirement set by us for standard operations is satisfied ere); otherwise it will not be a solution.

A question arises, whether the standard operations can be hosen arbitrarily. Formally speaking, they can. But non- ormally only those operations are selected as standard (solu- ions to any problems will have to be expressed through hem!) which are encountered in attacking numerous, or at east important, problems. The four arithmetical and some ther operations, such as the involution and taking the ictorial (the latter is discussed below, see Sec. 6), fall xactly into this class. Were the 'H'-operation necessary or a great number of problems, or were the cross-roads roblem of primary importance, the operation 'H' would e worth-while assigning to the set of standard ones. How- ver, the 'H'-operation did not deserve the honour yet and ardly ever will. We shall consider in Sec. 4 an operation, milar to 'H', which appears worthy to be included into ie set of standard operations.

---

[1]) Litzman, V. in his book *Giants and Dwarfs in the World of Num- rs*. Moscow, Fizmatgiz, (1959), gives as the highest of the calculated owers of 2 in a decimal notation—the number 2^{400}.

Let us go back now to our original problem concerning the crossings of the thousandth row. Its solution can be sought for in three different forms corresponding to the three foregoing interpretations of the term 'solution':

(1) as a line of 1001 numbers presented in a decimal notation; we shall not be seeking a solution in this form (this is beyond our reach: we cannot even find the solution in this form for only one crossing of the fourth row);

(2) in the form of an expression which in principle allows to determine for each crossing of the thousandth row the number (i.e. the decimal notation of this number) of people coming to it; we have already found such a solution: H^k_{1000}, the process of calculations being specified by Eqs. (1.1)-(1.4);

(3) in the form of an expression which not only enables us to calculate H^k_{1000} for any k from 0 to 1000 but also is composed of certain standard operations; this is the form in which we are going to obtain the solution; further exposition will show which operations are to be considered standard.

Section 3. Pascal's triangle

Let us consider an arbitrary line of numbers $d_0, d_1, \ldots, d_n$, $n = 0, 1, 2, \ldots$ (for $n = 0$ this line 'degenerates' into a line consisting of a single number d_0). Let us form out of this line a new line of numbers $s_0, s_1, \ldots, s_{n+1}$ according to the following rule:

$$s_0 = d_0, \tag{3.1}$$

$$s_k = d_{k+1} + d_k \ (1 \leqslant k \leqslant n), \tag{3.2}$$

$$s_{n+1} = d_n. \tag{3.3}$$

We shall refer to this line as to the one obtained from the preceding line *by the Pascal law*[1]). For instance, from the line 2, 0, —2 we generate by using Pascal's law the line 2, 2, —2, —2, and from the last one in its turn 2, 4, 0, —4, —2.

[1]) Pascal Blaise (1623-1662)—a great French scientist. Among other themes, Pascal investigated the properties of a triangular numerical table in which each line is obtained from the preceding one according to] the law specified by Eqs. (3.1)-(3.3). This table which we shall consider below was given the universally accepted name: 'Pascal's triangle'. Therefore the law on which generation of this table is based will be referred to as *the Pascal law*, and its lines—as *the Pascal lines*.

Note 1. If the line β is obtained from the line α by the Pascal law, then the sum of the terms of the line β is equal to twice the sum of the terms of the line α. Indeed, if Eqs. (3.1)-(3.3) are satisfied, then

$$s_0 + s_1 + s_2 + \ldots + s_n + s_{n+1} = d_0 + (d_0 + d_1) + \\ + (d_1 + d_2) + \ldots + (d_{n-1} + d_n) + d_n = \\ = 2(d_0 + d_1 + \ldots + d_n). \quad (3.4)$$

Note 2. Let us call a line of numbers $d_0, \ldots, d_n$ *symmetrical* if for any integer k from 0 to n the following equality is satisfied:

$$d_k = d_{n-k}. \quad (3.5)$$

The line of numbers $s_0, \ldots, s_{n+1}$, produced by the Pascal law from a symmetrical line $d_0, \ldots, d_n$ is itself symmetrical. To ascertain this we have to check the equality

$$s_k = s_{(n+1)-k} \quad (3.6)$$

for $k = 0, 1, \ldots, n+1$. But for $k = 0$ and $k = n+1$ the equality (3.6) follows from Eqs. (3.1)-(3.3) and from the equality $d_0 = d_n$ (obtained from Eq. (3.5) for $k = 0$). And if $1 \leqslant k \leqslant n$ we obtain

$$s_k = d_{k-1} + d_k = d_{n-(k-1)} + d_{n-k} = d_{(n+1)-k} + \\ + d_{[(n+1)-k]-1} = d_{[(n+1)-k]-1} + d_{(n+1)-k} = s_{(n+1)-k}. \quad (3.7)$$

Let us consider now the line consisting of a single number, unity. Let us call this line the *Pascal zero line*. Now we shall form out of this line another one which we shall call the *first Pascal's line*. The second Pascal's line is then obtained out of the first line, etc. Since the transformation to each next line adds unity to the number of terms in the preceding line, the nth Pascal's line will contain $n+1$ numbers. Without any calculations, just taking into account Notes 1 and 2, we can state that

(1) the sum of the numbers in the nth Pascal's line is equal to 2^n (because in each transformation from one line

to the next one the sum of the numbers is doubled, and for the zero line the sum is $2^0 = 1$);

(2) all the Pascal lines will be symmetrical (because the property of symmetry is retained in a transition from one line to the next, and the zero line is symmetrical).

Let us put down all Pascal's lines, beginning with the zero, so that each number of every line is placed below and between those two numbers of the preceding line whose sum makes this number. We shall obtain then an infinite table also called the *arithmetic Pascal's triangle*, as well as simply the *arithmetic triangle* or *Pascal's triangle*. When we look at the table, it appears to fill the inside of a certain angle; any of its initial parts formed by 0th, 1st, . . ., nth lines is triangular in shape. On p. 19 we give the initial part of the Pascal triangle formed by its first 15 lines, from zero to fourteenth.

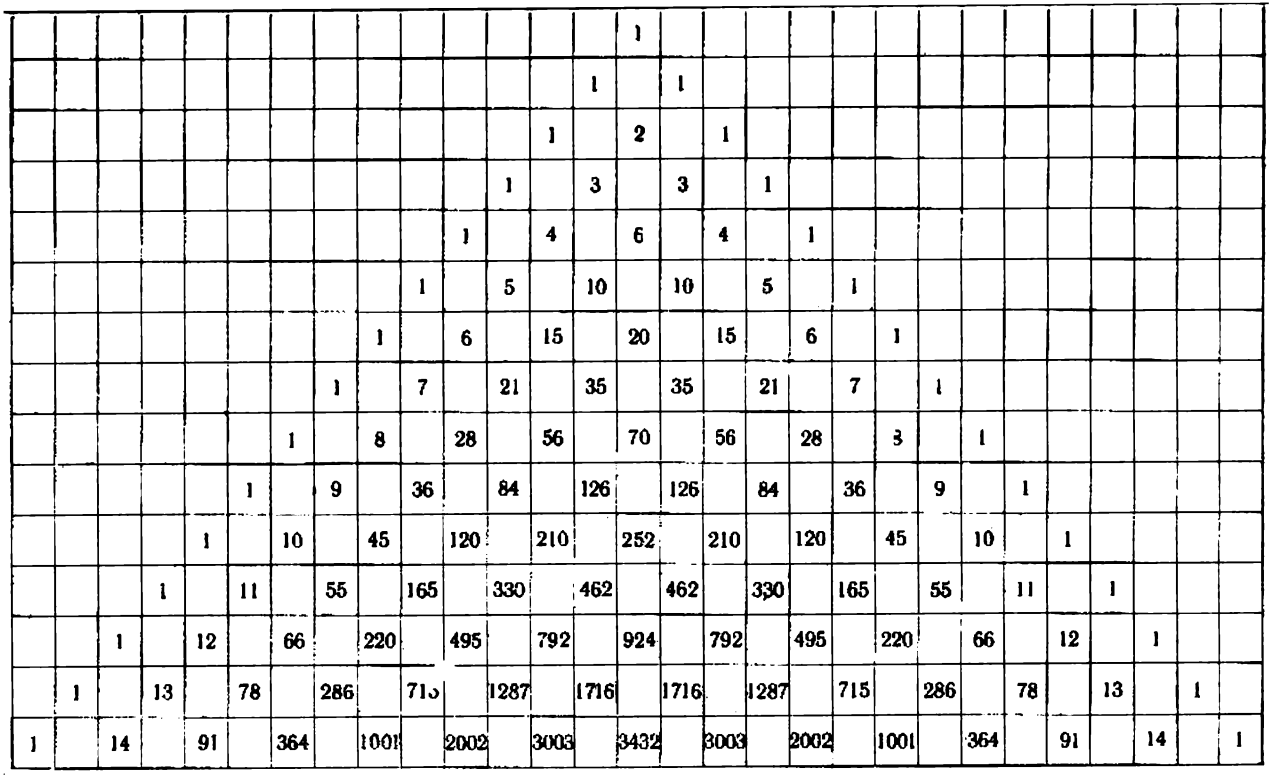
The Pascal triangle is symmetrical with respect to its bisector. The numbers filling it possess a number of interesting properties (for instance: the sum of the squares of the terms of any line is always equal to a number present in the same triangle; for any prime number p all members of the pth line, with the exception of the first and the last members, are divisible by p)[1]).

It is apparent that the method of generating the Pascal triangle could be formulated without resorting to the concepts of the 'Pascal law' or the 'Pascal line': Pascal's triangle is simply an infinite numerical table, 'triangular in shape', in which the sides are formed of unities and any number, excepting these unities, is generated as the sum of two numbers in the row above, one at its left and the other at its right. This was the form in which the triangle first appeared in Pascal's *Traité du triangle arithmétique* (Treatise on arithmetic triangle), published in 1665, after the author's death. To be more precise, this book contained the following table[2]) in which each number A is equal to the sum of two numbers: one preceding A in the same hori-

---

[1]) Some of these properties are discussed on pp. 89-73 and 103-106 of the book: Dynkin, E. B. and Uspensky, V. A. *Mathematical Discussions* (Matematicheskiye besedy). Moscow-Leningrad, Gostekhizdat (1952).

[2]) See Pascal, B. *Oeuvres complètes*, t. III. Paris, Hachette et Cie (1908).



zontal line and the other one preceding A in the same vertical column:

```
1 1  1  1   1   1  1  1 1 1
1 2  3  4   5   6  7  8 9
1 3  6 10  15  21 28 36
1 4 10 20  35  56 84
1 5 15 35  70 126
1 6 21 56 126
1 7 28 84
1 8 36
1 9
1
```

Therefore, our 'Pascal's triangle' differs from the 'triangle' discussed by Pascal himself by the 45°-rotation.

Pascal has investigated the properties and applications of his 'triangle' in great detail; some of these applications will be discussed in the next Section. Here we want to present only three properties of the triangle found by Pascal

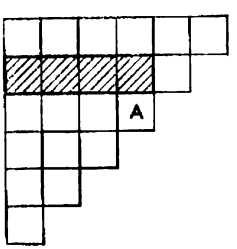


Fig. 3

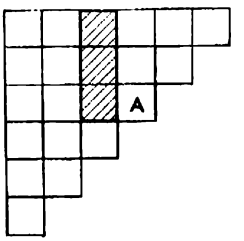


Fig. 4

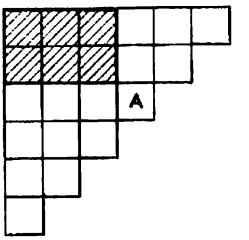


Fig. 5

himself; besides, in this Section (and only in this place of our exposition!) we shall base the discussion on the arrangement of the 'triangle' in a plane which was indicated by Pascal, and we shall refer to *horizontal rows* and *vertical columns*.

Property 1. Each number A in the table is equal to the sum of numbers in the preceding horizontal row, beginning with the extreme left and up to the one located directly above the number A (see Fig. 3 in which the cells containing the terms whose sum gives A, are shown hatched).

Property 2. Each number A in the table is equal to the sum of numbers of the preceding vertical column, beginning with the uppermost number down to the one adjacent to the number A on the left (Fig. 4).

Property 3. Each number A in the table, if reduced by unity, is equal to the sum of numbers filling the rectangle bounded by the horizontal row and the vertical column at the intersection of which A is located (these rows are not included into the rectangle in question) (Fig. 5).

We leave it for the reader to prove these propositions by himself (indication: the third property is easily derived from the first two).

However, a century before Pascal's treatise was printed, the table we are interested in (though not in a 'triangular' but in a 'rectangular' form) was already published in the *General Treatise on Number and Measure* (1556-1560) written by an outstanding Italian mathematician Niccolo Fontana, known as "Tartaglia" (the Stammerer), and which, similarly to Pascal's work, was partially published only after the death of the author. Tartaglia's table looked as follows[1]):

1	1	1	1	1	1
1	2	3	4	5	6
1	3	6	10	15	21
1	4	10	20	35	56
1	5	15	35	70	126
1	6	21	56	126	252
1	7	28	84	210	462
1	8	36	120	330	792

Here the upper line consists of unities; the number at the left end of each line is unity, and each of the next numbers is generated by adding two numbers, the one directly above and the other one adjacent to it from the left. It would be logical to call the table suggested by Tartaglia, the 'Tartalia rectangle'.

The members of each Pascal's line are usually enumerated from left to right, beginning with the zero number. Thus, the fourth position in the fifth line is occupied by the number 3. We shall denote by T_n^k the number occupying the kth position in the nth line, so that, for example, $T_0^0 = 1$, $T_5^4 = 10$, $T_{14}^4 = 1001$. Obviously, the expression T_n^k is defined for any $n \geqslant 0$ and $k = 0, 1, \ldots, n$.

Let us consider an infinite sequence formed by the numbers T_n^k at an arbitrary fixed k and changing n, i.e. the

[1]) See Zeuthen, H. G. *Geschichte der Mathematik im XVI und XVII Jahrhundert*, B. G. Teubner, Leipzig (1903).

sequence

$$T_k^k, \quad T_{k+1}^k, \quad T_{k+2}^k, \ \ldots, T_n^k, \ \ldots \tag{3.8}$$

The members of this sequence are the numbers which we find in Pascal's triangle in 'the kth line, counting from left to right, the line being parallel to the left side of the angle', and also, due to the symmetry of the triangle, the numbers of 'the kth line on the right parallel to the right-hand side of

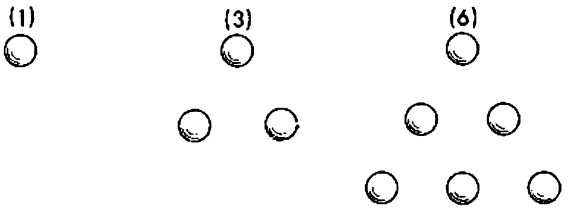


Fig. 6

the angle'. In Tartaglia's rectangle these numbers fill the kth column and kth horizontal row.

For $k = 0$ we obtain the sequence

$$1, 1, 1, 1, 1, 1, \ldots,$$

which consists of unities only (the zero line or zero column in Tartaglia's rectangle).

For $k = 1$ we obtain the series of natural numbers

$$1, 2, 3, 4, 5, 6, \ldots$$

(the first horizontal line or the first column of Tartaglia's rectangle).

For $k = 2$ we obtain the sequence

$$1, 3, 6, 10, 15, 21, \ldots$$

(the second line, or the second column of Tartaglia's rectangle). The members of this sequence are called the *triangular numbers*: 1—the first triangular number, 3—the second triangular number, 6—the third, etc; the mth triangular number is equal to T_{m+1}^2. They were given such a name because these numbers indicate the number of spheres or of other

identical objects laid out in a triangular pattern (see Fig. 6); in particular, the mth triangular number indicates how many members of Pascal's triangle are contained in its first m lines, viz. from the zero to the $(m-1)$ th.

For $k = 3$ we obtain the sequence

$$1, 4, 10, 20, 35, 56, \ldots$$

(the third line, or the third column of Tartaglia's rectangle). The members of this sequence are called *pyramidal numbers*,

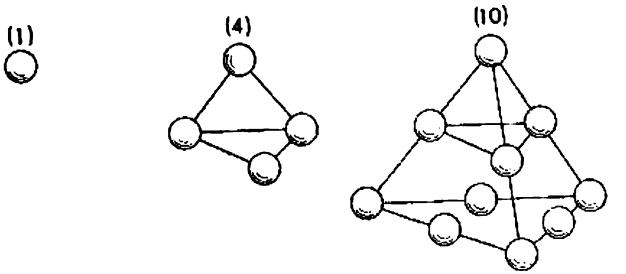


Fig. 7

or, more correctly, *tetrahedral numbers*: 1 is the first tetrahedral number, 4 is the second one, 10—the third, etc., so that the mth tetrahedral number is equal to T^3_{m+2}. These numbers indicate how many spheres can be built into a tetrahedron (Fig. 7)[1]).

---

[1]) Triangular and pyramidal numbers (which are particular cases of so-called figurate numbers) have aroused interest already in Ancient Greece where mystic properties were attributed to them. The earliest of the treatises that survived till our time and in which such numbers are considered, is the *Introduction to Arithmetics* by Nicomachos of Gerasa, the Classical Greek mathematician who lived about 100 A. D. (see Struik, D. J. *A Concise History of Mathematics*, vol. 1, Dover, N.Y. (1948); van der Waerden, B. L. *Science Awakening* (Engl. transl.), Groningen, Noordhoff (1954); Zeuthen, H.G. *Die Mathematik im Alterum und im Mittelalter*, Leipzig-Berlin, Teubner (1917).

However, indirect data indicate that multiangular numbers were known much earlier, in the II century B. C. (see van der Waerden, B. L., ibid) and even earlier than that, in the V century B. C., to Pythagoras and his students, pythagoreans (see van der Waerden, B. L., ibid., and Zeuthen, H. G., ibid., Struik, D. J., ibid.).

Section 4. Pascal's operation

By virtue of their definition the numbers T_n^k are subject to the following relationships:

$$T_0^0 = 1, \tag{4.1}$$

$$T_{n+1}^0 = T_{n+1}^{n+1} = 1 \quad \text{for} \quad n = 0, 1, 2, \ldots, \tag{4.2}$$

$$T_{n+1}^k = T_n^{k-1} + T_n^k \quad \text{for} \quad n = 0, 1, 2, \ldots,$$

$$k = 1, 2, \ldots, n. \tag{4.3}$$

These relationships completely define the numbers T_n^k; any number of lines in Pascal's triangle can be constructed by using Eqs. (4.1)-(4.3). The definition of the expression T_n^k can be extended in a natural manner so that it would be meaningful at any non-negative integer n and for any integer k. To achieve this we shall assume that $T_n^k = 0$ if $n \geqslant 0$ and k is such that at least one of the following two inequalities $0 \leqslant k$ and $k \leqslant n$ is violated. Therefore, $T_n^k = 0$ for all pairs (n, k) for which $n \geqslant 0$, $k < 0$, and for all pairs (n, k) for which $n \geqslant 0$, $k > n$. Now the relationship $T_{n+1}^k = T_n^{k-1} + T_n^k$ will hold for all k's (and not only for k from 1 to n, as is the case in Eq. (4.3)), and the numbers T_n^k will be completely defined by the following expressions:

$$T_0^0 = 1, \tag{4.4}$$

$$T_0^k = 0 \qquad \text{for} \quad k \neq 0, \tag{4.5}$$

$$T_{n+1}^k = T_n^{k-1} + T_n^k \text{ for all } n \geqslant 0 \text{ and all } k. \tag{4.6}$$

The foregoing arguments enable us to construct the following clear-cut picture of the generation of Pascal's triangle. Let us consider an infinite table composed of zeros arranged, as is shown below, in a checker-board order

. . . . . 0 0 0 0 . . . . .
. . . . . 0 0 0 0 0 . . . . .
. . . . . 0 0 0 0 . . . . .
. . . . . 0 0 0 0 0 . . . . .
. . . . . .

Obviously, *Pascal's law* will hold for this table which states that each number is the sum of two nearest to it numbers of the preceding line[1]).

Let us assume now that one of the zeros in the initial line of this table is substituted by unity. If we require for Pascal's law to be still valid, then the 'perturbation' will 'propagate angle-wise' like the waves from a stick thrust into a stream, i.e. as Pascal's triangle:

. . . . . 0 0 1 0 0 . . . . .
. . . . . 0 0 1 1 0 0 . . . . .
. . . . . 0 1 2 1 0 . . . . .
. . . . . 0 1 3 3 1 0 . . . . .
. . . . . .

Having chosen arbitrary n and k ($n = 0, 1, 2, \ldots$; $k = 0, 1, \ldots, n$) one can, provided he has enough time and patience, determine T_n^k. To do this we have to start writing out our Pascal's triangle and continue doing it until we reach the kth number in the nth line. Or we can just as easily make use of Eqs. (4.1)-(4.3); these equations make it possible to calculate T_n^k in a finite number of additions.

Problem (to be solved independently by the reader). How many addition operations must be performed to calculate T_n^k by means of Eqs. (4.1)-(4.3)? (Try to minimize the number of additions by using the symmetry properties of Pascal's triangle.)

Let us agree to call the operation consisting in determining the numbers T_n^k corresponding to the numbers n and k,

---

[1]) We shall assume that an infinite line

$$\ldots, s_{-3}, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots$$

is obtained from an equally infinite line

$$\ldots, d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots$$

according to Pascal's law, provided $s_k = d_{k-1} + d_k$ for any k. Then the definition of Pascal's law for finite lines is derived from the above assumption if each finite line

$$x_0, x_1, \ldots, x_n$$

is assumed to be identical to the infinite line

$$\ldots, 0, 0, x_0, x_1, \ldots, x_n, 0, 0, 0, \ldots$$

the *Pascal operation*. Pascal's operation is defined for any n and k for which $n \geqslant 0$, $0 \leqslant k < n$[1]).

And if we extend the definition of T_n^k in a manner described above on p. 24, then Pascal's operation will be defined for any non-negative integer n and for any integer k.

Pascal's operation enables us to express readily the numbers H_n^k which give the solution to the Olympiad problem presented in Sec. 1. In order to find such a form of expression we assume (for $m = 0, 1, \ldots, 1000$; $q = 0, 1, \ldots, m$)

$$Z_m^q = \frac{1}{2^{1000-m}} H_m^q, \tag{4.7}$$[2])

so that

$$H_m^q = 2^{1000-m} Z_m^q. \tag{4.8}$$

Then we obtain from Eqs. (4.7) and (1.1) that

$$Z_0^0 = \frac{1}{2^{1000}} \cdot 2^{1000} = 1. \tag{4.9}$$

Let us substitute the numbers H_m^q in Eqs. (1.2), (1.4) and (1.3) by their expressions via Z_m^q according to Eq. (4.8). From Eq. (1.2) we obtain

$$2^{1000-(n+1)} Z_{n+1}^0 = \frac{2^{1000-n} Z_n^0}{2},$$

whence

$$Z_{n+1}^0 = Z_n^0. \tag{4.10}$$

In the same way we obtain from Eq. (1.4)

$$2^{1000-(n+1)} Z_{n+1}^{n+1} = \frac{2^{1000-n} Z_n^n}{2},$$

whence

$$Z_{n+1}^{n+1} = Z_n^n. \tag{4.11}$$

---

[1]) However, Pascal himself (on the basis of the arrangement of the table in a plane suggested by him; see p. 20) was discussing in his treatise another operation, namely, the operation of finding for numbers x and y the number located at the intersection of the xth vertical column and the yth horizontal row (the rows are enumerated beginning with the 1st, so that this operation is defined for $x \geqslant 1$, $y \geqslant 1$). If we denote the sought number by $P(x, y)$, then it is easily demonstrated that $P(x, y) = T_{x-1+y-1}^{x-1}$, whence $T_n^k = P(k+1, n-k+1)$.

[2]) Since two to the power of zero is assumed equal to 1, for $m = 1000$ Eq. (4.7) takes the form $Z_{1000}^q = H_{1000}^q$.

And finally, from Eq. (1.3) we obtain

$$2^{1000-(n+1)}Z_{n+1}^{k}=\frac{2^{1000-n}Z_{n}^{k-1}+2^{1000-n}Z_{n}^{k}}{2},$$

whence

$$Z_{n+1}^{k}=Z_{n}^{k-1}+Z_{n}^{k}. \tag{4.12}$$

Equations (4.10)-(4.12) indicate that each line

$$\omega_{n+1}=\langle Z_{n+1}^{0},\ \ldots,\ Z_{n+1}^{n+1}\rangle,$$

where $n=0,\ 1,\ \ldots,\ 999$, is obtained from the preceding line

$$\omega_{n}=\langle Z_{n}^{0},\ \ldots,\ Z_{n}^{n}\rangle$$

according to Pascal's law. Since the initial line

$$\omega_{0}=\langle Z_{0}^{0}\rangle$$

is, as can be seen from Eq. (4.9), the zero Pascal's line, the line next to it, ω_1, is the first Pascal's line, the line ω_2 is the second Pascal's line, etc.; for each m from 0 to 1000[1]) the line ω_m is the mth Pascal's line, and

$$Z_{m}^{q}=T_{m}^{q}. \tag{4.13}$$

Hence, according to Eq. (4.8), for each $m=0,\ 1,\ \ldots$ $\ldots,\ 1000$ and for each $k=0,\ 1,\ \ldots,\ m$

$$H_{m}^{q}=2^{1000-m}T_{m}^{q}. \tag{4.14}$$

In particular,

$$H_{1000}^{q}=T_{1000}^{q}. \tag{4.15}$$

Thus, the number of people arriving at the crossings of the thousandth row are exactly the members of the thousandth Pascal's line. If Pascal's operation is regarded as a standard operation, then Eq. (4.15) supplies the solution to the problem discussed in Sec. 1 (in the third interpretation given at the end of Sec. 2). In the next two Sections we shall see how the solutions to two important problems can be found by means of Pascal's operation.

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[1]) The line ω_m is not defined for $m>1000$.

Section 5. Binomial coefficients

In this Section we shall demonstrate how the so-called *binomial coefficients* are expressed by means of Pascal's operation. The binomial coefficients are defined in the following way.

Let us take a binomial $1 + x$ and raise it to powers 0, 1, 2, 3, etc., arranging the resulting polynomials in the order of increasing powers of x. We shall obtain

$$(1 + x)^0 = 1, \tag{5.1}$$

$$(1 + x)^1 = 1 + x, \tag{5.2}$$

$$(1 + x)^2 = (1 + x)(1 + x) = 1 + 2x + x^2, \tag{5.3}$$

$$(1 + x)^3 = (1 + x)^2(1 + x) = 1 + 3x + 3x^2 + x^3, \tag{5.4}$$

etc.

In a general case, for any non-negative integer number n

$$(1+x)^n = a_0 + a_1x + a_2x^2 + \ldots + a_px^p, \tag{5.5}$$

where $a_0, a_1, \ldots, a_p$ are certain numbers. It would not be difficult to prove here that $p = n$ and $a_0 = a_p = 1$. However, just now this is not necessary yet. This result will follow from a more general formula somewhat later. Suffice it to say at this stage that the result of raising the binomial $(1 + x)$ to the nth power (where n is a non-negative integer number) can be written in the form of a polynomial arranged in the order of increasing powers of x and with numerical coefficients, as is shown by Eq. (5.5). The polynomial in the right-hand side of this equation is called the binomial expansion for the exponent n. Its coefficients (and their number is p) are, of course, dependent on n. To emphasize this dependence special notations are often used for these coefficients which include n. Namely, the coefficient at x^k in the binomial expansion for the exponent n is denoted by $\binom{n}{k}$. The numbers $\binom{n}{k}$ are called *binomial coefficients.*

Eqs. (5.5) can now be rewritten in the form

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{p}x^p, \tag{5.6}$$

and from Eqs. (5.1)-(5.4) we obtain

$$\binom{0}{0}=1$$

$$\binom{1}{0}=1 \quad \binom{1}{1}=1$$

$$\binom{2}{0}=1 \quad \binom{2}{1}=2 \quad \binom{2}{2}=1$$

$$\binom{3}{0}=1 \quad \binom{3}{1}=3 \quad \binom{3}{2}=3 \quad \binom{3}{3}=1.$$

We see that the lines of binomial coefficients for exponents $n = 0, 1, 2, 3$ coincide with the 0th, 1st, 2nd and 3rd lines of Pascal's triangle, respectively. We shall now prove that this is true for any n. To this end let us see how the line of coefficients for the exponent $n + 1$ is produced from the corresponding line of coefficients for the exponent n. Let us use the formula

$$(1+x)^{n+1}=(1+x)^n(1+x). \tag{5.7}$$

Let us write out for the left- and right-hand sides of this formula the corresponding expansions in the order of increasing powers of x. For the left-hand side, if we substitute n by $n + 1$, Eq. (5.6) takes the form

$$(1+x)^{n+1}=\binom{n+1}{0}+\binom{n+1}{1}x+\ldots+\binom{n+1}{k}x^k+\ldots$$
$$\ldots+\binom{n+1}{q}x^q, \tag{5.8}$$

where q is an arbitrary number. By virtue of the same equation (5.6) we obtain for the right-hand side:

$$(1+x)^n(1+x)=\left[\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{p}x^p\right](1+x)=$$
$$=\binom{n}{0}+\binom{n}{1}x+\ldots+\binom{n}{k}x^k+\ldots+\binom{n}{p}x^p+$$
$$+\binom{n}{0}x+\ldots+\binom{n}{k-1}x^k+\ldots+\binom{n}{p-1}x^p+\binom{n}{p}x^{p+1}=$$
$$=\binom{n}{0}+\left[\binom{n}{0}+\binom{n}{1}\right]x+\ldots+\left[\binom{n}{k-1}+\binom{n}{k}\right]x^k+\ldots+$$
$$+\left[\binom{n}{p-1}+\binom{n}{p}\right]x^p+\binom{n}{p}x^{p+1}. \tag{5.9}$$

By virtue of Eq. (5.7) the right-hand sides of Eqs. (5.8) and (5.9) are equal. Therefore $q = p + 1$; by equating the coef-

ficients at equal powers of x we obtain

$$\binom{n+1}{0} = \binom{n}{0} \tag{5.10}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k} \tag{5.11}$$

$$\binom{n+1}{p+1} = \binom{n}{p}. \tag{5.12}$$

The relationships (5.10)-(5.12) indicate that the line of coefficints in the expansion for the exponent $n + 1$ is obtained from the line of expansion coefficients for the exponent n according to Pascal's law. Since the line of coefficients for the exponent 0 coincides with the zero Pascal's line, all the subsequent lines of coefficients will also coincide with the respective lines of Pascal's triangle. Hence, the numbers $\binom{n}{k}$ are defined only for $k = 0, 1, \ldots, n$, and also

$$\binom{n}{k} = T_n^k. \tag{5.13}$$

Note. The reply to the question 'With what coefficients are x^{-3} and x^{20} found in the binomial expansion for the exponent 5?' can be 'with zero coefficients'. Therefore it appears natural to extend the definition of the expression $\binom{n}{k}$ to the cases $k < 0$ and $k > n$, assuming in these cases $\binom{n}{k} = 0$. Then, by virtue of the extended definition of the symbol T_n^k accepted in the preceding Section, Eq. (5.13) will be valid for all non-negative integers n and all integers k.

Thus, we have expressed the binomial coefficients by applying Pascal's operation. Now Eq. (5.6) can be rewritten as follows:

$$(1+x)^n = T_n^0 + T_n^1 x + T_n^2 x^2 + \ldots + T_n^k x^k + \ldots + T_n^n x^n. \tag{5.14}$$

The formula (5.14) is sometimes called *Newton's binomial formula* or simply *Newton's binomial*[1]). We shall give in Sec. 7 another, more traditional representation of this formula.

We can say that in the present Section we were considering the following 'problem of binomial coefficients': to find the expression for $\binom{n}{k}$. As we know from Sec. 2, there may be different interpretations of what is accepted as a solu-

[1]) The formula (5.14) was known long before Newton—and to above-mentioned Tartaglia, among others. Newton's name was associated with the formula in question only because in 1676 he pointed to a method of generalizing this formula for the case of arbitrary rational (including negative) exponents.

tion to the problem of this type. If, for example, we consider as a solution an expression which makes it possible to calculate for the given n and k the corresponding binomial coefficient (in a decimal notation) then $\binom{n}{k}$ itself will be such a solution. We shall require that the solution express $\binom{n}{k}$ via the numbers n and k and standard operations; this interpretation of the solution will depend on the choice of standard operations. If Pascal's operation is a postulated standard, then the formula (5.13) will give the solution to the problem of binomial coefficients. In Sec. 7 we shall give another solution of this problem, i.e. the solution for another set of standard operations.

Section 6. The number of parts of a given set

In mathematics any ensemble of objects is called a *set*. For example,

(a) association of all pages of this booklet;

(b) association of all integers;

(c) association of all even numbers;

(d) association of all pencils in any given box—all of them are sets.

If an object and a set are indicated, two situations may realize:

(1) the object under consideration belongs to the set in question;

(2) the object under consideration does not belong to this set.

In the first case the object in question is said to be an element of the considered set. For instance, the number 3 is an element of the set of all integers and is not an element of the set of all odd numbers.

It may happen that all elements of a set A are at the same time the elements of another set B (for example, all elements of the set of all even numbers are the elements of the set of all integers). In this case the set A is called a *part*, or a *subset* of the set B. Obviously, each set is a subset of itself. If the set A is a subset of the set B and the set B is a subset of the set A, this means that A and B consist of identical elements (i.e. they are 'one and the same set').

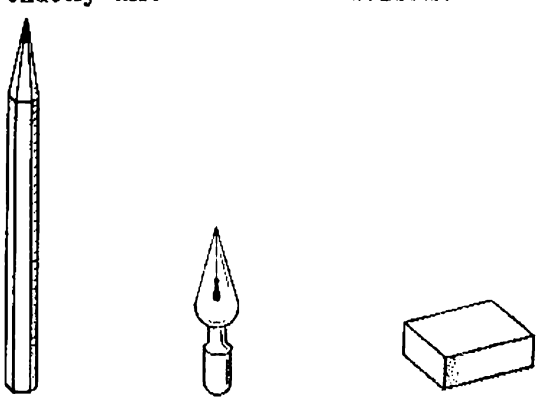
Sets may be *finite* (like sets in the above examples (a) and (d)) or *infinite* (like sets in the above examples (b) and (c)). Finite sets (and in this Section we are to deal exclusively with such sets) make the subject of a special branch of mathematics, called *combinatorics*.

One special set occupies a unique position among finite sets, viz. a set containing no elements; such a set is called a *null set*, or an *empty* set. For instance, one cannot exclude the possibility that having opened the box we shall find that the set of pencils it contains is empty. Here is what P. S. Aleksandrov said about an empty set[1]):

'Among finite sets we consider also an empty set, i.e. the set which contains no elements; the number of elements of the empty set is zero. The necessity of considering the empty set is apparent from the fact that when we define in one way or another a set, we cannot know in advance whether it contains at least one element or not. It is probable, for example, that the set of ostriches which at this moment are to the North of the North Pole is empty; however, we cannot rely on this statement with certainty since there is a chance that a ship captain had brought an ostrich into the region in question'. The empty set is considered as a subset of any set.

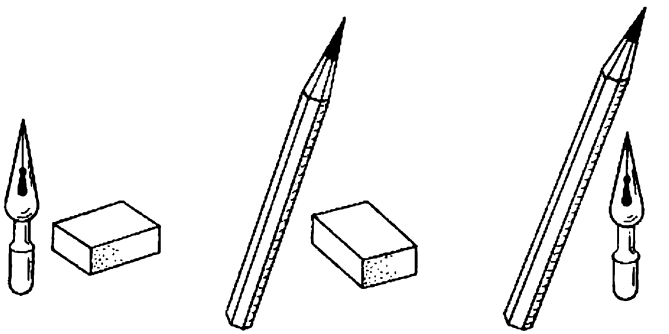
If a set is finite, its elements can be counted, thereby the number of its elements will be found. A set consisting of n elements is called the 'n-element set'. For instance, the set of pages of this booklet is the 87-element set; the empty set is the zero-element set.

Example. Let us consider a set consisting of three objects: a pencil, a pen and an eraser. Let us identify all its subsets. There is exactly one zero-element subset: an empty set. There are exactly three one-element subsets:

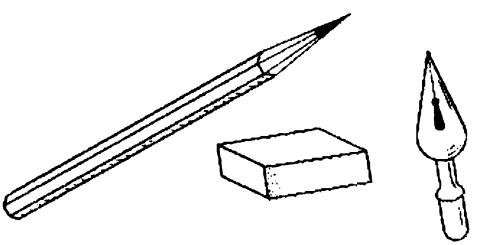


[1]) Aleksandrov, P. S. *Introduction into the General Theory of Sets and Functions.* Moscow-Leningrad, Gostekhizdat (1948).

There are exactly three two-element subsets:



And, finally, there is exactly one three-element subset (coinciding with the whole set):



All in all, our set thus has eight subsets.

Let a set be given consisting of n elements. Any of its k-element subsets is called a *combination of n specified elements taken k at a time.* (These are the very combinations studied in the school course on algebra in Section "Conjunc-

tions"). The number of combinations of n specified elements taken k at a time is evidently independent of exactly which elements were specified but depends only on the numbers n and k. Therefore this number is called shortly the *number of combinations of n elements taken k at a time*, and is denoted by

$$C_n^k.$$

In other words, C_n^k is the number of k-element subsets of an n-element set. The expression C_n^k is usually regarded as meaningful for $n = 0, 1, 2, \ldots, 0 \leqslant k \leqslant n$[1]).

We shall denote the number of all subsets of an n-element set by C_n, so that

$$C_n = C_n^0 + C_n^1 + \ldots + C_n^n. \tag{6.1}$$

But what are the numbers C_n, C_n^k equal to? We can answer some of these questions immediately. Thus, it follows from the example discussed above that $C_3 = 8$, $C_3^0 = C_3^3 = 1$, $C_3^1 = C_3^2 = 3$.

Furthermore, the following properties hold:

The first property of the number of combinations:

$$C_m^0 = C_m^m = 1. \tag{6.2}$$

Proof. Indeed, an m-element set E has exactly one 0-element subset (empty set) and exactly one m-element subset (the set E itself).

Let us establish now two more properties of numbers C_n^k, without calculating the numbers themselves. To prove the second property will be useful from the standpoint of grasping the ideas presented in this Section; as for the third property, it will be this one which, together with the first, will constitute the basis for the calculation of numbers C_n^k.

The second property of the number of combinations:

$$C_n^k = C_n^{n-k}. \tag{6.3}$$

Proof. Let us take an arbitrary set M of n elements. We have to prove that the number of its k-element subsets is equal to the number of its $(n - k)$-element subsets. Let us carry out the following experiment imaginary. We cut from

[1]) It is logical to consider, though, that the expression C_n^k is meaningful also for $k > n$ and in that case is equal to zero (since in this case there are no k-element subsets at all).

a sheet of paper as many squares as there are k-element subsets in our set (i.e. C_n^k) and draw on each of them one of these subsets, so that each k-element subset be drawn on one of the squares. Let us then cut from paper C_n^{n-k} circles and draw each of the $(n-k)$-element subsets on exactly one of these circles. Now it will be sufficient to demonstrate that the number of squares is the same as that of circles. To do this we shall lay all the squares on the table and on each one of them place a circle by the following rule: if a square displays a certain k-element subset of the set M,

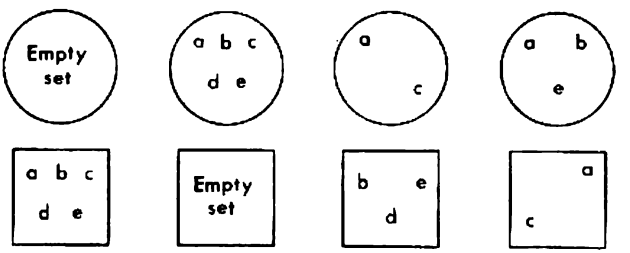


Fig. 8

then we must place on it a circle displaying a subset of the set M consisting of the remaining $n-k$ elements, i.e. of all those elements which are not comprised in the subset shown on the square (Fig. 8 shows several squares together with corresponding to them circles for the case of a five-element set M composed of elements a, b, c, d, e). It is evident that each square will be covered by exactly one circle, and each circle will be placed on exactly one square. This means that the number of circles is equal to the number of squares.

Before passing over to the third property let us prove the following

Lemma. *Separate an arbitrary element in an $(n+1)$-element set. The number of k-element subsets of this set incorporating the separated element is equal to C_n^{k-1}.*

Proof. Let us again perform an imaginary experiment with circles and squares. We cut out of a paper sheet as many squares as there are k-element subsets containing the selected element, and draw on each of them one of these subsets so that each subset be shown. Now we cut out C_n^{k-1}

paper circles and on each one of them draw one of the $(k-1)$-element subsets of the set composed of all not selected elements, in order that all such subsets be shown (the number of non-selected elements is n, and so the number of such subsets is exactly C_n^{k-1}). Now we start covering each square by a circle according to the following rule: if a square displays a certain set A then we must cover it with a circle displaying the set produced from the set A by removing the separated element. Obviously, each square will be covered by exactly one circle, and each circle will be placed exactly over one square. This means that the number of squares is equal to that of circles, i.e. to C_n^{k-1}. But we were cutting out exactly as many squares as there are k-element subsets in an $(n+1)$-element set containing the separated element. Hence, the number of such subsets is equal to C_n^{k-1}, Q.E.D.

Now we go over to the third property of the number C_n^k.

The third property of the number of combinations:

$$C_{n+1}^k = C_n^{k-1} + C_n^k. \tag{6.4}$$

Proof. Let us take an arbitrary $(n+1)$-element set M and compose all its k-element subsets. Select now one of the elements out of M, and denote it by the letter a. Let us denote by X the number of those k-element subsets of the set M which include the element a, and by Y—the number of those k-element subsets of the set M which do not include a. Then

$$C_{n+1}^k = X + Y. \tag{6.5}$$

But by virtue of the lemma, $X = C_n^{k-1}$. As for Y, it is the number of combinations of n non-selected elements taken k at a time, i.e. C_n^k. Therefore

$$C_{n+1}^k = C_n^{k-1} + C_n^k, \tag{6.6}$$

Q.E.D.

The third property, taken together with the first, demonstrates that the line

$$C_{n+1}^0, \quad C_{n+1}^1, \ \ldots, C_{n+1}^{n+1} \tag{6.7}$$

is generated from the line

$$C_n^0, C_n^1, \ \ldots, C_n^n \tag{6.8}$$

according to Pascal's law. Since for $n = 0$ the line

$$C_0^0 \tag{6.9}$$

coincides, due to Eq. (6.2), with the zero Pascal's line, then for an arbitrary n the line (6.8) will coincide with the nth Pascal's line; therefore,

$$C_n^k = T_n^k. \tag{6.10}$$

Thus, we are able to calculate the number of k-element subsets of an n-element set, viz. the number of combinations of n elements taken k at a time (Eq. (6.10) yields, as a result, the solution to the problem 'The number of combinations' provided that Pascal's operation is considered standard[1])).

As for the number of all subsets of an n-element set, Eqs. (6.1) and (6.10) demonstrate that this number is equal to the total sum of members of the nth Pascal's line, and as we already know, this sum is equal to 2^n. The final result is then

$$C_n = 2^n. \tag{6.11}$$

Section 7. The relation to factorials

In Sec. 4 two methods were given for calculating the number T_n^k from the specified n and k: a more 'mechanical' (though leading to superfluous computations) method of gradual writing out Pascal's triangle, and a more economical from the standpoint of the number of operations (though requiring a certain procedure in calculations) method which consists in applying Eqs. (4.1)-(4.3). Both these methods are very similar and in fact both follow directly from the mothod of generating the numbers T_n^k by Pascal's law. However, there is also an absolutely different way of finding T_n^k. We shall proceed to expose it.

First of all let us introduce one symbol. We set

$$0! = 0,$$

and for each integer m

$$m! = (m - 1)!m.$$

[1]) The reader will find a different solution, obtained for a different set of standard operations, in Sec. 7.

Therefore, for $m > 0$

$$m! = 1 \times 2 \times \ldots \times m.$$

The expression $m!$ is read: 'factorial of the number m', or more briefly: 'm factorial'.

Now let us express Pascal's operation in terms of taking the factorial and arithmetic operations. To achieve this, consider the following expression:

$$\frac{m!}{q!\,(m-q)!}.$$

We denote this expression by F_m^q. Obviously, the expression F_m^q is meaningful for $m \geqslant 0$, $0 \leqslant q \leqslant m$. We note that

$$F_0^0 = \frac{0!}{0!\,0!} = 1.$$

Furthermore,

$$F_m^0 = \frac{m!}{0!\,m!} = 1, \quad F_m^m = \frac{m!}{m!\,0!} = 1.$$

And finally

$$F_n^{k-1} + F_n^k = \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{k!\,(n-k)!} =$$

$$= \frac{n!}{(k-1)!\,(n-k)!\,(n-k+1)} + \frac{n!}{(k-1)!\,k\,(n-k)!} =$$

$$= \frac{n!}{(k-1)!\,(n-k)!}\left[\frac{1}{n-k+1} + \frac{1}{k}\right] =$$

$$= \frac{n!}{(k-1)!\,(n-k)!} \cdot \frac{n+1}{k\,(n-k+1)} = \frac{(n+1)!}{k!\,(n+1-k)!} = F_{n+1}^k.$$

Thus the line

$$F_0^0$$

is the zero Pascal's line, and the '$(n+1)$'th line

$$F_{n+1}^0,\ F_{n+1}^1,\ \ldots,\ F_{n+1}^{n+1}$$

is produced from the 'n'th line

$$F_n^0,\ F_n^1,\ \ldots,\ F_n^n$$

according to Pascal's law. Therefore for any $m = 0, 1, 2, \ldots$ the line

$$F_m^0, F_m^1, \ldots, F_m^m$$

coincides with the mth Pascal's line, and

$$F_m^q = T_m^q.$$

Hence,

$$T_m^q = \frac{m!}{q!(m-q)!}.$$

Thus we have expressed Pascal's operation via the operations of taking the factorial, subtraction, multiplication and division, in the sense that we have found for T_m^q an expression which only contains along with q and m the symbols of the enumerated operations. This enables us to calculate T_m^q provided we can calculate factorials, differences, products and quotients.

The following corollaries can be derived from the formula established above for T_m^q:

Corollary 1. By reducing the numerator and the denominator in the derived expression for T_m^q by $(m-q)!$ we obtain

$$T_m^q = \frac{m(m-1)\ldots[m-(q-1)]}{1, 2 \ldots q}.$$

Corollary 2. Let $m \geqslant 1$, $1 \geqslant q$. The product of q cofactors $m(m-1)\ldots[m-(q-1)]$ is always divisible by the product of q factors $1 \cdot 2 \ldots q$.

By virtue of the Corollary 1 the ratio of these two products is equal to T_m^q, and T_m^q is an integer.

Corollary 3. From. Eq. (4.15) it follows that

$$H_{1000}^q = \frac{1000!}{q!(1000-q)!}$$

This is a new form of the solution to the problem from Sec. 1.

Corollary 4. From Eq. (5.13) we obtain

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\ldots[n-(k-1)]}{1 \cdot 2 \ldots k}.$$

This is a traditional 'school' expression for a binomial coefficient.

Corollary 5. From Eq. (5.14) and Corollary 1 we find

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \ldots + \\ + \frac{n(n-1)\ldots[n-(k-1)]}{1\cdot 2\ldots k}x^k + \ldots + x^n.$$

This is the traditional 'school' form of the binomial theorem.

Corollary 6. Eq. (6.10) yields a traditional 'school' form for the number of combinations

$$C_n^k = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\ldots[n-(k-1)]}{1\cdot 2\ldots k}.$$

CERTAIN APPLICATIONS OF MECHANICS TO MATHEMATICS

Translated from the Russian

by A. Repiev

Section 1. The problem of a tangent to a circumference

It is well known that a *tangent* to a circumference is the straight line that has only one point in common (called the *point of tangency*) with this circumference. In text-books on geometry it is proved that a tangent is perpendicular to the radius drawn from the center of the circumference to the point of tangency. The following is the proof of this theorem based on mechanical concepts.

Let us perform a thought experiment. Suppose that a weight is attached to the end of a string M and hangs from the string, the other end being fixed to a point O. Common experience leads the reader to assert that

(A) *The weight occupies the lowest possible position it can take being attached to the string.*

This fact, which is of decisive importance for our consideration, is self-evident; it might be instructive, however, to observe that it is a particular case of a more general law (formulated within the so-called "principle of potential energy minimum"), which we are to deal with in Sec. 4 in a more complex situation. To be more correct, with the help of the above-mentioned "principle of potential energy minimum" Statement (A) is deduced from the following evident Statement (E):

(E) *There exists but one equilibrium position for the weight,* i.e. the position of the weight at est is completely determined by the position of the point O and the length of the string.

It follows further from the uniqueness of the equilibrium (Statement E) that

(C) *The weight is on the vertical line drawn from the point of suspension.*

In fact, if the weight were not on this vertical, then turning the string with the weight about that vertical, we would obtain a new equilibrium (Fig. 1). Thus, the above reason-

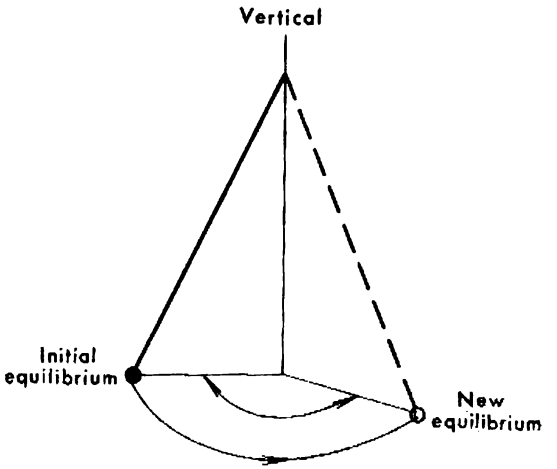


Fig. 1

ing is based on one more fact known from practice, i.e.

(S) *If a body or a system of bodies being in equilibrium is turned about a vertical, then a new position will also be an equilibrium one.*

Statements (A) and (C) suggest that

(B) *The distance from the weight to the point of suspension O is equal to the length of string MO* (meaning that the string is tight).

[It can be noted paranthetically that Statements (B) and (C) appear to be not less evident than those from which they are deduced.]

Now we may proceed to prove our theorem. Let O be the center of a circumference (Fig. 2), and a straight line p be tangent to the circumference at P. It is to be proved that $OP \perp p$. Let us reproduce our drawing on a vertical wall, so that the straight line p would be horizontal and the circumference disposed wholly *over* that line p. (If this book is put vertically upward, Fig. 3 will meet the above requirements.) Note that the point P will now be the lowest point of the circumference. Let us next take a string as long as the radius of the circumference, and fix one end at the center O of

the circumference. To the other end designated by M a weight will be attached and suspended from the string. Let us show that the end M will fall on point P. Indeed, first, by virtue of (B), the end M will lie only on the circumference; second, by virtue of (A), this end will be at the lowest point of that circumference, i.e. at P. Thus, the string will coincide with

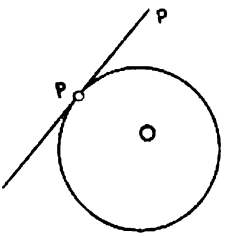


Fig. 2

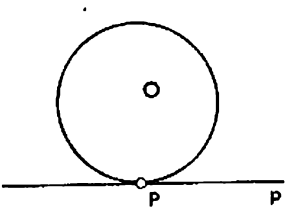


Fig. 3

radius OP. So, by virtue of (C), that radius is perpendicular to the straight line p, which completes the proof.

The example considered above is not very interesting of course as it involves the proof of a well known and rather simple theorem. This very method, however, will be employed in proving new theorems. In doing so we will make use of the "mechanical" proof of the theorem of a tangent to a circumference. In the following section we will primarily concern ourself with a natural generalization of this theorem, namely, the theorem of a tangent to an ellipse.

Section 2. The problem of a tangent to an ellipse

A circumference is defined as the locus of points A such that their distance AO to a given point O is equal to a pre-as-

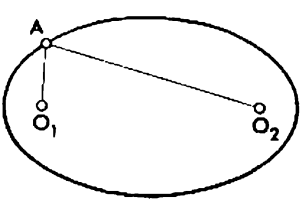


Fig. 4

signed value l. The generalization of this definition is that of an ellipse. The *ellipse* (Fig. 4) is the locus of points A

such that the sum of their distances $AO_1 + AO_2$ to two given points O_1 and O_2 is equal to a pre-assigned value l. The points O_1 and O_2 are called *foci* of the ellipse, and l—the *length of the major axis* of the ellipse. Segments AO_1 and AO_2 connecting any point A on the ellipse with the foci are called *focal radii* through the point A. A circumference is a particular case of an ellipse when points O_1 and O_2 coincide. In that case the focal radii also coincide and are equal to the radius of the circumference.

Anybody can draw an ellipse if he takes a string with both ends fixed at certain points O_1 and O_2 on a sheet of

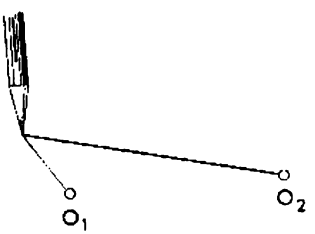


Fig. 5

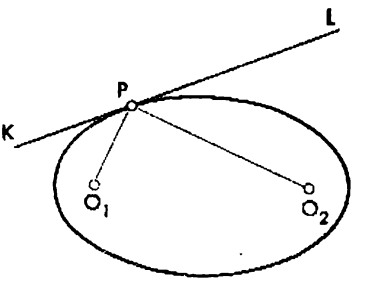


Fig. 6

paper in such a way that the distance O_1O_2 is shorter than the length of the string and draws a line pulling the string with a tip of a pencil (Fig. 5)[1]).

As in the case of the circumference, a straight line is said to be a *tangent* to an ellipse if it has one common point with the ellipse (Fig. 6).

The following **theorem of a tangent to an ellipse** is valid. *A tangent to an ellipse makes equal angles with focal radii drawn to the point of tangency* (in Fig. 6 such angles are KPO_1 and LPO_2).

A particular case of this theorem is the theorem of a tangent to a circumference. In fact, with a circumference both focal radii coincide and therefore the theorem of a tangent to an ellipse for the case of circumference is thus: a tangent forms equal adjacent angles with the radius drawn to the

[1]) About properties of ellipse see *Remarkable Sines* (Introduction to the Theory of Elliptic Functions) by Markushevich, A. I.

point of tangency, which implies perpendicularity of the radius and the tangent.

Now we will provide a proof of the theorem of a tangent to an ellipse analogous to that given for the theorem of a tangent to a circumference. The reader is recommended, before proceeding further to read the text, to invent such a "mechanical" proof of his own.

As in the previous section, prior to the proof we shall perform a mechanical experiment. Let us fix two points $\bar{O}_1$ and $\bar{O}_2$ on a vertical plane at an equal height. We take a string longer than the distance $\bar{O}_1\bar{O}_2$ and attach the ends of the string to the points $\bar{O}_1$ and $\bar{O}_2$. A weight is fitted on

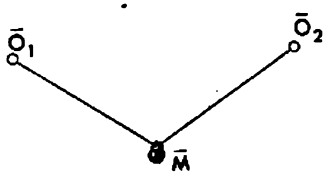


Fig. 7

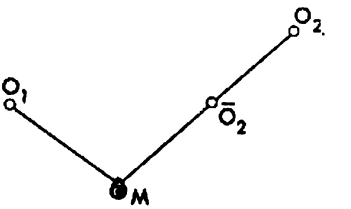


Fig. 8

the string in such a manner that it can slide along the latter (e.g. we pass the string through an ear of a weight as shown in Fig. 7). Now if we let the weight to hang free on the string, it eventually will appear at point $\bar{M}$ where it will stay at rest. Two following facts are obvious:

($\bar{C}_1$) *The plane* $\bar{O}_1\bar{O}_2\bar{M}$ *is vertical* (i.e. it passes through a vertical).

($\bar{C}_2$) *Angles formed by the segments* $\bar{M}\bar{O}_1$ *and* $\bar{M}\bar{O}_2$ *with the horizontal that passes through the point* $\bar{M}$ *in the plane* $\bar{M}\bar{O}_1\bar{O}_2$ *are equal.*

Note that Statements ($\bar{C}_1$) and ($\bar{C}_2$) can be inferred from the statement of the uniqueness of the equilibrium position of a weight. For this purpose draw a vertical line through the mid-point of the segment $\bar{O}_1\bar{O}_2$ and turn our string with the weight about that straight line through 180°. The end $\bar{O}_1$ of the string will now be at the point $\bar{O}_2$, and the end $\bar{O}_2$ at the point $\bar{O}_1$. According to Statement (S) in the preceding section, we obtain a state of equilibrium, therefore, by

virtue of the supposed uniqueness of this state, the string (together with the triangle $\overline{M}\overline{O}_1\overline{O}_2$) will coincide with its previous position. Since the triangle turned 180° coincides with its former self, its axis of rotation lies in the plane of the triangle. The plane $\overline{M}\overline{O}_1\overline{O}_2$ thus passes through the vertical axis, i.e. it is vertical. Further, in the turning, the horizontal line through $\overline{M}$ in the plane $\overline{M}\overline{O}_1\overline{O}_2$ will coincide with its former self, and an angle formed by this horizontal with the segment $\overline{M}\overline{O}_1$ will coincide with an angle made by that line with the segment $\overline{M}\overline{O}_2$. Hence, these angles are equal.

We now drop the assumption that the points of suspension are at one level. Let O_1 and O_2 be arbitrary points. We suspend from them a weight sliding freely along the string (Fig. 8). Let the weight at point M be at rest. Note that if the weight is at point M and if we fix the string at any point, then the position of the weight will not change. [In fact, if, say, we fix the string at point $\overline{O}_2$, (see Fig. 8) the pull of segment $\overline{O}_2O_2$ of the string will be substituted by the reaction of the support at point $\overline{O}_2$.] The above fact warrants the extension to the general case of conclusions made in consideration of points of suspension at different levels. Actually, it is sufficient to fasten the string at point $\overline{O}_2$ located at the same height as O_1 (Fig. 8[1])); the equilibrium position will therewith be the same. Putting $\overline{O}_1 = O_1$, we have the case of level suspension points $\overline{O}_1$ and $\overline{O}_2$ considered above. Thus from $(\overline{C}_1)$ and $(\overline{C}_2)$ we immediately obtain Statements $(\overline{C}_1)$ (self-evident by itself) and $(\overline{C}_2)$:

(C_1) *The plane O_1O_2M is vertical.*

(C_2) *The angles formed by segments MO_1 and MO_2 and the horizontal straight line drawn through the point M in the plane MO_1O_2 are equal.*

At last, valid are two apparent statements:

(A) *In equilibrium the weight takes the lowest possible position it can occupy being fixed to the string* (by means of the principle of potential energy minimum mentioned in Sec. 1, this statement follows from the uniqueness of the position of equilibrium).

[1]) In the drawing O_2 is higher than O_1, the case where O_1 is higher than O_2 is considered in an analogous way.

(B) *In equilibrium the string is tight* (which means that segments MO_1 and MO_2 are straight, and hence the sum of distances from the point M to O_1 and O_2 is equal to the length of the string).

Let us return to the proof of the theorem of a tangent. Let it be an ellipse with foci O_1 and O_2 (see Fig. 6) with the length of major axis equal to l and the straight line KL tangent at the point P. It is to be proved that $\angle O_1PK = \angle O_2PL$. To prove, turn the drawing so that the plane

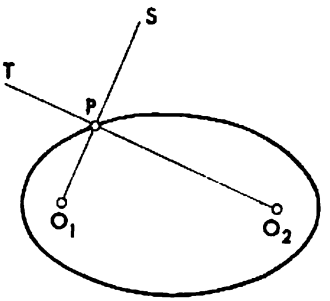


Fig. 9

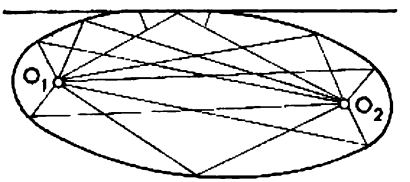


Fig. 10

of the drawing is vertical, the straight line KL is horizontal and the ellipse is over the straight line KL (then P will be the lowest point of the ellipse). Now we take a string of a length l, fit a weight to it and fix the ends of the string to the points O_1 and O_2, and then let the weight free. It will occupy a certain position M. Then point M will, by virtue of (C_1), lie in the plane of the drawing. By virtue of (B), it will lie on the ellipse. Moreover, it will, by virtue of (A), coincide with P. Thus the string will coincide with the segments PO_1 and PO_2. By virtue of (C_2), angles formed by those segments and the straight line KL will be equal.[1])

The property of the tangent to an ellipse to form equal angles with focal radii enables one to construct with a pair of compasses and a ruler a tangent to an ellipse at a given point (provided its foci are given). For this, it is sufficient to draw straight lines through the point of tangency P and foci O_1 and O_2 (see Fig. 9) and find the bisector of the angle

[1]) The student will find another proof of this theorem also utilizing the concepts of mechanics in *Variational Problems* (Sec. 11) by L. A. Lyusternik.

SPO_2 (or angle TPO_1). This bisector will be the required tangent.

The theorem of tangent to an ellipse has one interesting optical interpretation: if we assume that the ellipse is a mirrow, i.e. rays of light travelling in the plane reflect from the ellipse, then the rays emitted by a point source located at one focus will collect at another focus (Fig. 10).

In fact, an angle formed by an incident ray and the ellipse is equal by the laws of optics to an angle formed by the reflected ray and the ellipse. An angle formed by a straight line and a curve (in our case by the ray of light and the ellipse) is measured as an angle between that straight line and the curve drawn through the vertex of the angle. Since the incident ray travels along the focal radius, the reflected ray by virtue of the theorem proved above, will travel along the other focal radius.

Section 3. The problems of tangents to a parabola and a hyperbola

The *parabola* (Fig. 11) is the locus of points P that are equidistant from a given point F (called the *focus* of the parabola) and a given straight line d (called the *directrix* of the parabola)[1]). A parabola divides the plane into two parts: located in one of them is the directrix, and in the other (dashed in Fig. 12) the focus. A straight line having just one point in common with the parabola and positioned in one of the parts into which the parabola divides the plane is called a *tangent* to the parabola. In Fig. 13 the straight line p is a tangent, and the straight line q is not a tangent, though it has only one common point with the parabola.

The following **theorem of tangent to parabola** is valid: *if from an arbitrary point P of the parabola* (Fig. 14) *we draw the segment PF connecting P with the focus and segment PD perpendicular to the directrix, then the angles made by these segments and the tangent p at the point P are equal* (so, in Fig. 14, $\angle DPK = \angle KPF$).

This theorem can be "proved" by "mechanical" considerations in a way similar to the theorems in two previous sections. We give a sketch of such a "proof".

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[1]) In more detail about parabola see Markushevich's book that has been already mentioned.

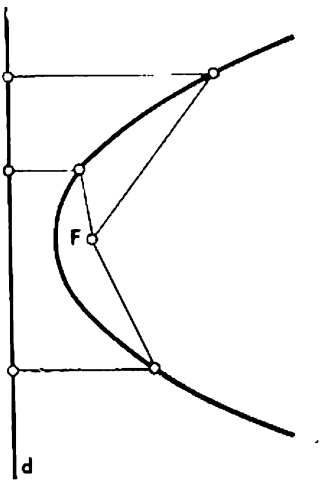

Fig. 11

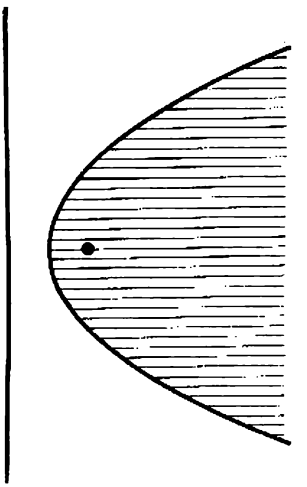

Fig. 12

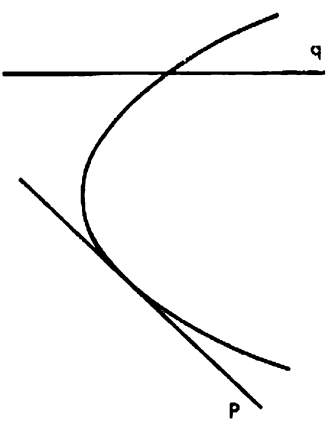

Fig. 13

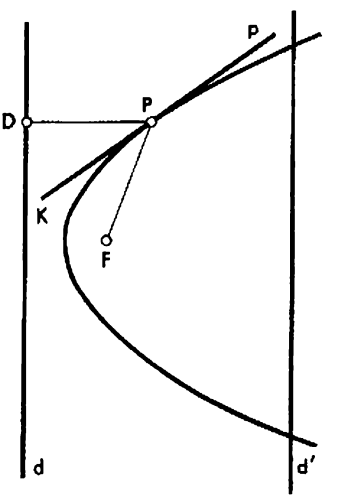

Fig. 14

Let us draw a line $d' \parallel d$ at an arbitrary dist
straight line d, so that point P is between d a
the drawing in such a way that it is vertical, th
is horizontal and the parabola is over p. We wi

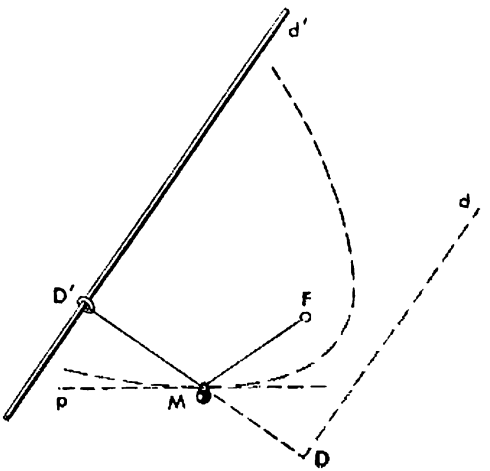

Fig. 15

the straight line d' to be a thin rod along which
(Fig. 15). We attach a string of length l to the r
other end of the string fix to the point F. On
a weight will be fitted, which can slide along
Let an equilibrium for the weight be M, the r
time being at D'. The string will be tight, th
segments MF and MD' of the string will be str
of all, note that

$$MD' \perp d'$$

(If it were not the case then as will be seen in
component T_1 of the force T of the string tensio
rod d' would be non-zero and make the ring shif
repeating the reasoning of the preceding sectio
first, that the angles formed by the segments M
and the straight line p are equal and, second, th

$$MD' + MF = l$$

Let us draw $MD \perp d$. Since the distance betwe
is l, from (3.1) and (3.2) it follows that

$$MD = MF$$

This means that M lies on the parabola. But since M takes the lowest possible position and P is the lowest point of the parabola, M will coincide with P.

Thus, in the final analysis, the angles which the segments PF and PD form with the straight line p, are equal.[1])

The optical interpretation of the proved theorem is as follows: the rays of light emitting from the focus of the

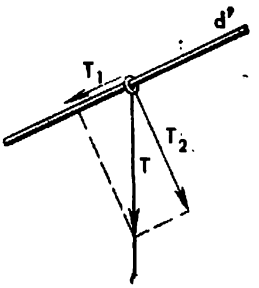

Fig. 16

parabola after reflecting from the latter become a pencil of rays perpendicular to the directrix (Fig. 17), a fact that is made use of in searchlights, reflectors and other devices

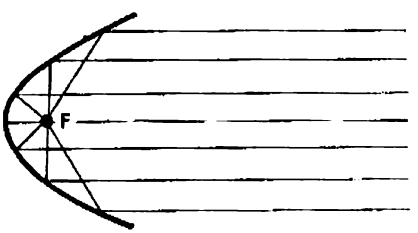

Fig. 17

serving to impart the rays (light or heat) the required direction; the reflecting surface of each of those devices being a surface of revolution of a parabola about a perpendicular dropped from the focus onto the directrix.

The *hyperbola* is a locus of points whose distances from two fixed points F_1 and F_2 (called *foci*) are constant[2]). The

[1]) Another "mechanical" proof of this theorem is also given in *Variational Problems* by L. A. Lyusternik.

[2]) About hyperbola see *Remarkable Sines* by A. I. Markushevich.

hyperbola, as seen in Fig. 18, consists of two branches. The distance of any point P' on the left branch (in Fig. 18) to F_1 is *less* than the distance from P' to F_2 by a given amount; the distance from any point P'' on the right branch to F_2 is *less* than the distance from P'' to F_1 by the same

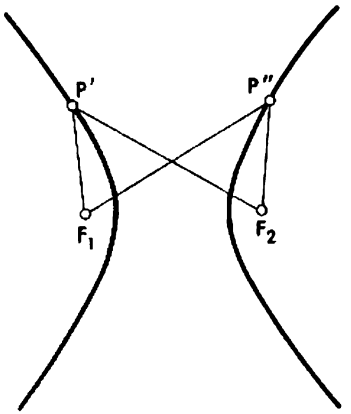

Fig. 18

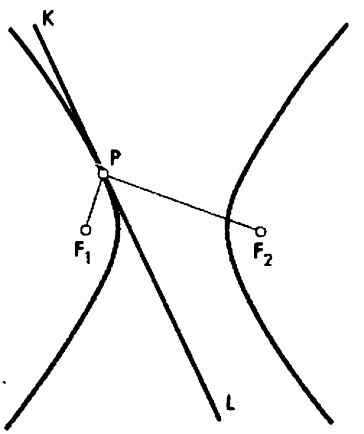

Fig. 19

amount. As with the parabola, each branch of the hyperbola divides the plane into two parts. The straight line entirely located within one of these parts and having just one point in common with that branch is called a *tangent to a given branch of the hyperbola*. Fig. 19 shows a straight line tangent to the left-hand branch of a hyperbola at point P. It can be shown that a tangent to one branch of the hyperbola will not intersect the other branch. Any tangent to a branch of a hyperbola is simply called a *tangent to the hyperbola*.

As in the case of the ellipse, the straight segments connecting the foci with a point on the hyperbola are called *focal radii*.

Theorem of a tangent to a hyperbola: *a tangent to a hyperbola makes equal angles with focal radii drawn to the point of tangency* (thus, in Fig. 19, $\angle F_1PL = \angle LPF_2$).

We will not prove the theorem (we leave it to the reader), but merely give the underlying "mechanical" construction. By the definition of the hyperbola, the difference between

two focal radii is a constant; let us designate this difference by a. Let the point of tangency P lie on the right-hand branch (Fig. 20). Draw from the left-hand focus F_1 a circumference with a radius r so large that, first, $r > a$, and, second, the

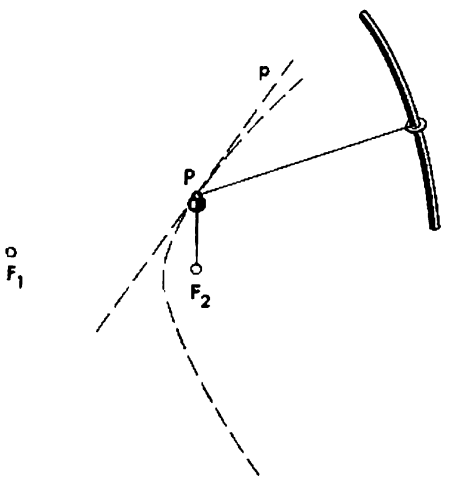

Fig. 20

point P lies inside that circumference. We picture the circumference as a wire ring with another smaller one sliding along the former.

We take a string of length r-a and pass it through a weight which can slide along it, fix one end of the string to the smaller ring, the other end being attached to the right-hand focus F_2. Now, if we turn the drawing so that the straight line p (the tangent) is horizontal and an appropriate branch of the hyperbola lies over the straight line p, then the weight on the string will be in equilibrium at point P. It is this fact that allows to prove the theorem by reasoning we have already used above. Note that if the point of tangency lies on the left branch, then the construction may be repeated describing the circumference from the right focus. In that case it is also possible to use the circumference with the center at the left focus, but then the string should be taken of length $r + a$.

As with the previous theorems of tangents, the theorem of tangent to hyperbola provides the way for construction of the latter.

Section 4. Principle of potential energy minimum

A weight raised to a certain height is capable of doing work in falling, i.e. possesses potential energy. As we know from the school course on physics, the potential energy of a weight q raised to a height h is determined by the product qh. It is seen that the lower the weight, the smaller is the potential energy. The tendency of the weight to take the lowest position is connected with the potential energy tending to decrease. When the weight is attached to a string, then in the lowest position it can occupy its potential energy will be minimal. Therefore, Statement (A), which played a decisive part in proving the theorems of tangents, i.e. asserting that in equilibrium a weight occupies the lowest possible position, is equivalent to the following statement:

(U) *In a position of equilibrium the potential energy of a weight attains its lowest value.*

In turn, Statement (U) [and hence (A)] is a corollary of Statement (E) on the uniqueness of the equilibrium position and of the following statement:

(D′) *If in a certain position the potential energy of a weight attains the lowest value, then this position is an equilibrium.*

To deduce (U) from this it suffices to observe that if the potential energy did not reach the lowest value, then there would, by virtue of (D′), exist some other equilibrium positions corresponding to the lowest value of the potential energy, but that would be inconsistent with Statement (E).

Statement (D′) is a particular case of the general principle of mechanics known as the principle of potential energy minimum, or the Dirichlet principle[1]), which says:

The position of the system in which the potential energy has its lowest value is a position of equilibrium.

If the equilibrium position is unique, the Dirichlet principle permits an important corollary:

In equilibrium the potential energy of the system reaches its lowest value.

The derivation of this corollary is analogous to the deduction of Statement (U).

Henceforth, we shall consider only those cases when the equilibrium is unique (also possible are other cases; so, shown in Fig. 21 are four equilibrium positions for a ball).

[1]) See the basis of this principle in Lyusternik's book (Sec. 13) that has been already mentioned.

Of course, to solve problems on tangents there is no need to recourse to the principle of potential energy minimum; that the weight takes the lowest possible position is self-evident. However, in other cases, when dealing with not a single but any number of weights bound together, it would be erroneous to claim that in equilibrium all of them occupy the lowest possible positions; here positions

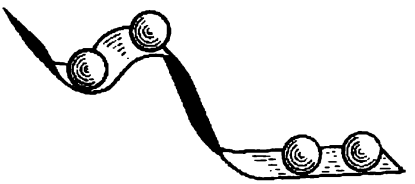

Fig. 21

of all the weights are to be found, for which purpose the concept of potential energy would often appear convenient.

The following is an example.

In a remarkable book by the Polish mathematician H. Steinhaus "Mathematical snapshorts" [Oxford University Press, 1960] there is such a problem (theme 39):

"A school is to be built for three villages. In the first village there live 50 children, in the second 70, and in the third 90. Where must the school be located so that the total time spent by the children in walking to it is minimal?

To solve this problem it is sufficient to lay the ground plan flat on a table (Fig. 22), then we drill holes in the table in the places of villages and pass three strings through the holes tying their upper ends in one bundle and the lower ends being provided with weights 50, 70 and 90 units, respectively (e.g. 500, 700 and 900 g). The school should be built where the knot will be located. (Why?)"

To answer this "why" calculate the potential energy of the system consisting of three weights. Let the weights be equal to q_1, q_2 and q_3, respectively, and if the first weight is at height h_1, the second at h_2, and the third at h_3, the potential energy is the sum of potential energies of separate weights[1]):

$$E = q_1h_1 + q_2h_2 + q_3h_3 \qquad (4.1)$$

[1]) In fact, the potential energy is measured by the work that can be done. If we cut the strings, each weight will do work equal to its potential energy and the whole system will do work equal to the sum of these separate works.

Now, let r_1, r_2 and r_3 be distances from the knot to the first, second and third villages on the map, respectively; l_1, l_2, l_3 be the lengths of the first, second and third strings, respectively, and h be the height of the table surface. Evidently, whatever the point at which the knot will fall, and

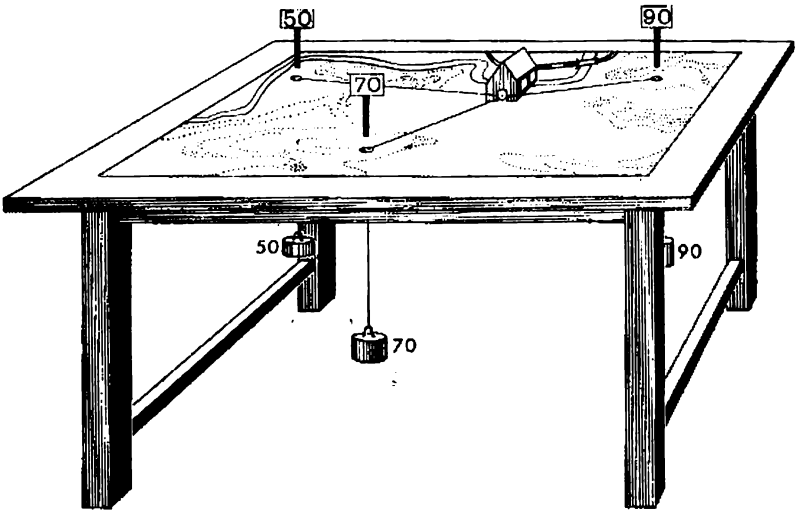

Fig. 22

whatever the heights h_1, h_2, h_3 at which the weights will be located, the relationships hold

$$r_1 + (h - h_1) = l_1; \quad r_2 + (h - h_2) = l_2; \quad r_3 + (h - h_3) = l_3$$

or

$$h_1 = r_1 + h - l_1; \quad h_2 = r_2 + h - l_2; \quad h_3 = r_3 + h - l_3$$

Then equality (4.1) can be written

$$E = q_1 r_1 + q_2 r_2 + q_3 r_3 + C$$

where

$$C = (q_1 + q_2 + q_3)\, h - q_1 l_1 - q_2 l_2 - q_3 l_3$$

is a constant independent of the position of weights.

By virtue of the principle of potential energy minimum, if E is attaining a minimum, the system is in equilibrium. Hence, on the assumption of equilibrium uniqueness it follows that in equilibrium the quantity E is minimal. But

if the quantity E is minimal, then the quantity

$$T = q_1 r_1 + q_2 r_2 + q_3 r_3 = E - C$$

is minimal as well. But T is exactly the total time required for the children to come from their villages to that school where the knot is located. So, with the knot position corresponding to the system of equilibrium this total time actually attains its lowest value.

Section 5. Particles and the center of gravity

So far, we did not concern ourselves with the dimensions of those weights suspended from strings. It is evident from the above considerations that, on the one hand, we assume the weights to be very small (so small that the position of each of them can be referred to as a point), on the other hand, we allotted them certain weights, and hence, masses.

The above considerations lead us to one of the basic concepts of mechanics—the concept of a particle. In mechanics, by a particle (or a mass point) is meant a body so small that its dimensions may be neglected. The particle may be thought of as a geometrical point with a certain number ("mass") assigned to it. Since the particle possesses a definite mass, it has a definite weight too.

Suppose there is a system of particles subjected to the action of the force of gravity proportional to the mass of a particle. The resultant of all these parallel forces is applied at a special point called the center of gravity of a given system of particles. The position of the center of gravity depends on the locations of the particles of the system and their masses.

In mechanics, any body is regarded as a collection of a great number of particles. Generally speaking, these particles have different masses, so that masses within the body are distributed irregularly, some parts of the body having larger masses, others smaller. We shall further assume that some parts of the body have no masses at all and are thereby "weightless". The center of gravity of the system of particles constituting the body is, by definition, the center of gravity of the body. If the mass of the body is concentrated in a finite number of individual particles (we shall deal with such cases below, e.g. a weightless rod with point weights fitted on it), the center of gravity of such

a body apparently coincides with that of the system of these individual particles.

Many properties of a body depend exclusively on the position of the center of gravity. As is known from elementary physics, the equilibrium of the body depends on the relative position of its center of gravity and a number of points at which it is fixed. Hence, the following statements are valid:

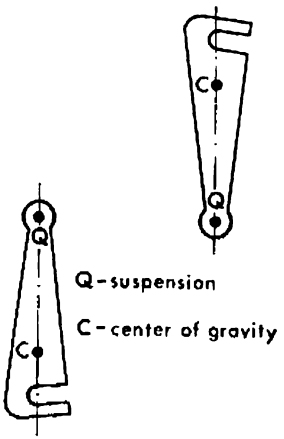

Fig. 23

Statement 1. *If a body, fixed at one point, is at rest, then that point and the center of gravity are on one vertical* (Fig. 23).

Statement 2: *If a body with a support surface or a knife edge is at rest, then a vertical dropped from the center of gravity traverses that surface or edge* (Fig. 24).

These Statements (which can be employed for the experimental determination of the center of gravity) suggest, in particular, the following: if a body is substituted by another having the same form and center of gravity, then the equilibrium position will not change. In a number of

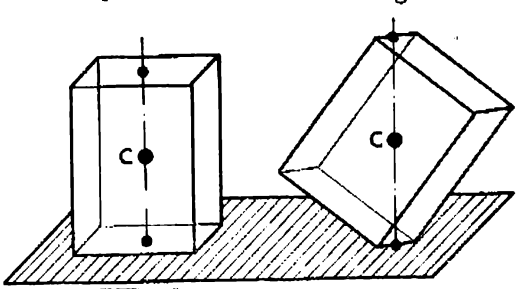

Fig. 24

cases, however, it is necessary to take into account not only the location of the center of gravity but the total mass of the body as well (i.e. the resultant mass of particles constituting the body). For example, if we replace not all the body but only a *part* of it by a new body with the same configuration and the same position of the center of gravity

as the removed part, then the position of the center of gravity of the *whole of the body* can change due to a change in the mass of that part. The knowledge of the total mass and the center of gravity is sufficient for dealing with many problems, at any rate, those that we are to encounter below. It is natural, therefore, to introduce the following definition of the equivalence of two systems of particles.

Two systems of particles are equivalent, if (1) the center of gravity of the first system coincides with the center of gravity of the second system, and (2) the total mass of all the particles of the first system is equal to that of all the particles of the second system.

The concept of equivalence of systems introduced above is remarkable for that in replacing a part of a system by a system equivalent to that part we will have a system equivalent to the initial one. Let us give this statement a more clear formulation.

Given a system $\bar{\alpha}$ of particles $\bar{M}_1, \ldots, \bar{M}_r$ with masses $\bar{m}_1, \ldots, \bar{m}_r$ and a system α of particles $M_1, \ldots, M_s$ with masses $m_1, \ldots, m_s$. We combine both these systems into one common system α which will be represented in the following way:

1° Include in α any particle of the system $\bar{\alpha}$ coinciding in location with none of the points of the system α.

2° Include in α any particle of the system α coinciding in location with none of the points of the system $\bar{\alpha}$.

3° If a certain particle $\bar{M}_i$ of the system $\bar{\alpha}$ coincides in location with a certain particle M_j of the system α then we shall include in α a new particle which coincides in location with particles $\bar{M}_i$ and M_j and whose mass is equal to the sum $\bar{m}_i + m_j$ of the indicated particles. The following is valid:

Statement 3. *If a system α is a result of amalgamation of systems $\bar{\alpha}$ and α and a system β be a result of amalgamation of systems $\bar{\beta}$ and β and if the systems $\bar{\alpha}$ and $\bar{\beta}$ are equivalent and the systems α and β are equivalent, then the systems α and β will be equivalent as well.*

The proof of Statement 3. It is apparent that the total masses of the systems α and β coincide. Let us show that the centers of gravity of these systems coincide as well. The resultant of the gravity forces acting on the particles

of system α can be obtained in such a way: first, find the resultant $\overline{P}$ of gravity forces acting on $\overline{\alpha}$, then find the resultant P of gravity forces acting on α and at last, find the resultant P of these resultants. In the same way the resultant Q for the forces of gravity acting on the system β can be obtained as a resultant of two forces, $\overline{Q}$ and Q, where $\overline{Q}$ is the resultant of forces acting on $\overline{\beta}$ and Q is the resultant of forces acting on β. By virtue of equivalence of $\overline{\alpha}$ and $\overline{\beta}$ the forces $\overline{P}$ and $\overline{Q}$ coincide. Similarly, the forces P and Q coincide. Therefore, the resultants of these forces — $\overline{P}$ and $\overline{Q}$, also coincide. In particular, the points of their application coincide as well, the points being the centers of gravity of systems α and β.

Note that for any system of particles an equivalent system can be constructed consisting only of one particle. To do this it is sufficient to consider a particle located at the center of gravity of the original system and having a mass equal to the total mass of the system. It is this particle that forms the system equivalent to the original one. The particle located in the center of gravity of any system and having a mass equal to the total mass of the system will be called the *material center* of this system. We can define equivalent systems as such systems the material centers of which coincide.

Section 6. The center of gravity of a two-particle system

Let two weights P and Q be fitted on a weightless rod, placed at a distance d from each other (Fig. 25). Find the

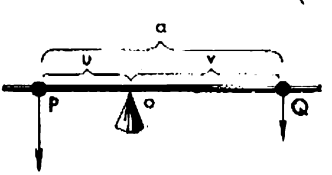

Fig. 25

center of gravity of the rod with the weights. By virtue of Statement 1 in the preceding section, our question is equivalent to the following: at what point O should the rod be supported so as to be in equilibrium? Denote the distances

of the sought point O from the weights P and Q by u and v. The equilibrium will be only when the product of force P by the arm u is equal to the product of the force Q by the arm v, i.e.

$$Pu = Qv \tag{6.1}$$

Comparing equality (6.1) with the equality

$$u + v = d \tag{6.2}$$

we obtain

$$u = \frac{Q}{P+Q}\,d, \quad v = \frac{P}{P+Q}\,d$$

Notice that $\frac{u}{v} = \frac{Q}{P}$, thus the distances from the center of gravity to the weights are in inverse proportion to the masses of the weights.

In particular, if both weights are equal in mass, then the center of gravity is midway between the weights (which is self-evident), conversely, if it is known that the center of gravity is in the middle, it follows that the two weights are equal.

If one of the weights is twice as heavy as the other, the center of gravity will be closer to the heavier one, the spacing between the weights being in ratio 1:2, conversely, if it is known that the center of gravity divides the spacing in the ratio 1:2, then one of the weights (namely, the one which is closer to the center of gravity) will be twice as heavy as the other.

Since the center of gravity of the rod, whose mass is lumped at two points M and N coincides with the center of gravity of the two particles M and N (with appropriate masses), we have arrived at the following result:

The center of gravity of a two-particle system lies on the straight line connecting those points. Its distance from the points is inversely proportional to their weights, and hence to their masses.

In particular, the center of gravity divides the distance between the points by halfs only when the two particles possess equal weights; the center of gravity divides the spacing between the particles in the ratio of 1:2 only when one of the particles will be twice as heavy as the other, the center of gravity being closer to the heavier particle.

Section 7. Theorems on the intersection of straight lines

Consider a system of particles α which constitutes a union of two systems γ and δ. Let M and N denote the material centers of systems γ and δ, respectively. By virtue of Statement 3 in Sec. 5 the system of particles M and N is equivalent to the system α. The material center of system α coincides, therefore, with that of the system of the particles M and N and thus lies on the straight line connecting

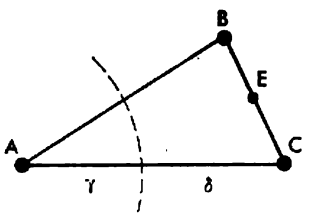

Fig. 26

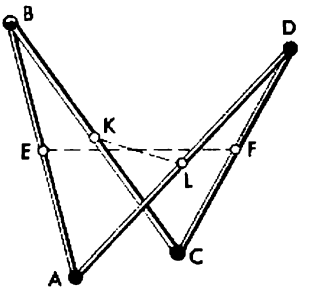

Fig. 27

these particles. Since the material center of the system is located at its center of gravity, we obtain the following theorem:

The center of gravity of a system of particles constituting a union of two systems γ *and* δ *lies in the straight line connecting the centers of gravity of systems* γ *and* δ.

Now we will supply three geometric applications of this theorem.

Medians of the triangle intersect at one point. In the vertices of a triangle we will position equal point weights, which will be denoted by the same letters A, B, C as the respective vertices (Fig. 26). The resultant system of particles will lbe divided into two parts γ and δ, so that A enters into γ and B and C into δ. The center of gravity of the system γ is at A, and the center of gravity of the system δ is located in the middle E of the side BC as follows from the results of the preceding section. From the theorem in the beginning of this section, the center of gravity O of the system A, B, C lies on the median AE. In a similar manner we determine that O lies on two other medians as well.

Thus, all the three medians are bound to intersect at one point.

As the material center M of system δ (located at E) is twice as heavy as the material center of system γ, which coincides with particle A, the center of gravity of the system $\{M, A\}$ divides the segment EA in the ratio 1 : 2, whereupon it is to be found closer to E. But this center of gravity is the one of the system A, B, C, i.e. the common point of intersection of all medians. We derived, therefore, a well known theorem that the median intersection point intercepts one third of each median as viewed from a respective side.

The space rectangle. Given a space rectangle $ABCD$ (Fig. 27). Show that the straight lines EF and KL connecting the mid-points of opposite sides intersect. Position equal

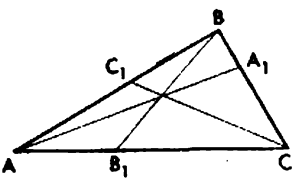

Fig. 28

weights at vertices of the rectangle, which are assumed to be particles. As with the "proof" of the theorem of medians it suffices to find that the center of gravity of the rectangle lies in each of the straight lines EF and KL. Let us show, for example, that it lies on EF (the fact that it lies on KL is to be determined in an analogous fashion). To accomplish this, divide the system of particles A, B, C, D into two parts; one of them includes the particles A and B and the other, C and D. The center of gravity for the first part will be the point E, for the second part the point F. From the theorem in the beginning of the section the center of gravity for the whole rectangle is located on the straight line EF. We leave it to the reader to see that the segments EF and KL, when intersecting, are divided by halves.

Ceva's theorem. The theorem that the medians of a rectangle interesect at one point is a particular case of the following proposition known as Ceva's theorem:

Let A_1, B_1, C_1 be three points lying in sides BC, CA and AB, respectively, of the triangle ABC (Fig. 28); *for the straight*

lines AA_1, BB_1, CC_1 to intersect at one point it is necessary and sufficient for the relationship

$$\frac{AB_1}{B_1C}\cdot\frac{CA_1}{A_1B}\cdot\frac{BC_1}{C_1A}=1 \tag{7.1}$$

to be valid.

The reader is recommended to verify that relationship (7.1) actually holds in the following cases:

1. The straight lines AA_1, BB_1 and CC_1 are bisectrices of the triangle. (*Note*: Use the theorem that a bisectrix of an angle in a triangle divides the opposite side into parts proportional to the adjacent sides.) It suggests that the bisectrices of the triangle intersect at one point.

2. The straight lines AA_1, BB_1, and CC_1 are heights of the triangle. (*Note*: Use the formula representing the lengths of the segments into which a height divides a respective side.) It suggests that heights of an acute triangle intersect at one point.

Let us proceed to prove the Ceva's theorem.

Let relationship (7.1) be valid. Let us show that the straight lines AA_1, BB_1, CC_1 intersect at one point. Reasoning proceeds in the same way as in the preceding two items. We will show that with appropriately selected weights located at vertices, the center of gravity of the system lies in each of the straight lines AA_1, BB_1, CC_1. Let (to a certain scale)

$$AB_1=a, \quad B_1C=b$$
$$CA_1=c, \quad A_1B=d$$
$$BC_1=e, \quad C_1A=f$$

Thus,

$$\frac{a}{b}\cdot\frac{c}{d}\cdot\frac{e}{f}=1 \tag{7.2}$$

At the vertex A we place a weight equal to bd, at the vertex B—a weight equal to ac, and at the vertex C—a weight equal to ad. The center of gravity of the system $\{A, C\}$ is located at a point such that its distances from the vertices A and C are inversely proportional to the weights at these vertices, but B_1 is such a point. Similarly, the center of gravity of the system $\{B, C\}$ lies at the point A_1. Lastly, the center of gravity of the system $\{A, B\}$ is located at a point, the distances of which x and y to the vertices A and B are inversely proportional to the weights

bd and ac, i.e.

$$\frac{x}{y}=\frac{ac}{bd} \quad \text{and} \quad x+y=f+e$$

But it results from relationship (7.2) that

$$\frac{ac}{bd}=\frac{f}{e}$$

Therefore, $x=f$ and $y=e$. Consequently, the center of gravity of the system $\{A, B\}$ is at the point C_1. We must note that by virtue of the theorem at the beginning of the section, the center of gravity of the triangle as a whole lies on each of the straight lines AA_1, BB_1, CC_1.

Conversely, let the straight lines AA_1, BB_1, CC_1 intersect at a single point O; we show that relationship (7.1)

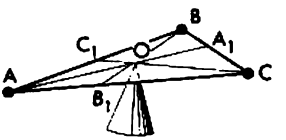

Fig. 29

is valid. For this, imagine our rectangle to be a weightless plate; lay it flat, support it at point O and attach weights at the vertices in such a way that the rectangle is in equilibrium (Fig. 29). By Statement 1 in Sec. 5, point O is the center of gravity for the triangle with the weights at the vertices selected and therefore it is the center of gravity of the whole system. Hence point O must lie on a straight line connecting A with the center of gravity of the system $\{B, C\}$; consequently, A_1 is the center of gravity of the system $\{B, C\}$. Likewise, B_1 and C_1 are the centers of gravity of the systems $\{C, A\}$ and $\{A, B\}$. If we denote the masses of the particles A, B, C as p, q, r, then according to the preceding section

$$\frac{AB_1}{B_1C}=\frac{r}{p}, \quad \frac{CA_1}{A_1B}=\frac{q}{r}, \quad \frac{BC_1}{C_1P}=\frac{p}{q}$$

In multiplying these proportions we obtain the required relationship (7.1).

Section 8. The center of gravity of a rod with weights

Let us summarize the considerations of Sec. 6.

Imagine a weightless rod (Fig. 30) with weights P_1, P_2, . . ., P_n and state the problem of determining the center of gravity of this rod. As in Sec. 6, to solve the problem it is sufficient to find a point O such that if the rod is supported at that point, it is in equilibrium. Calculate the abscissa x of such a support point [1]).

Let the abscissa of the weight P_1 be equal to a_1, of weight P_2 to a_2, etc.; lastly, let the abscissa of the weight P_n be equal to a_n. Suppose (Fig. 30) that the supporting point O

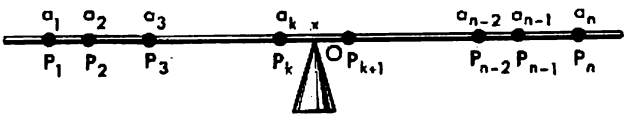

Fig. 30

is between the weights P_k and P_{k+1} (we do not exclude the case when one of the weights is located exactly at O). Then, the arms for the forces P_1, . . ., P_k will be $x - a_1$, ..., $x - a_k$, respectively, and the arms for the forces P_{k+1}, . . ., P_n will be $a_{k+1} - x$, . . ., $a_n - x$. Since the rod is in equilibrium, the sum of moments of forces tending to turn it counterclockwise will be equal to the sum of moments of forces tending to turn it clockwise, i.e.

$$P_1(x-a_1)+\ldots+P_k(x-a_k)=$$
$$=P_{k+1}(a_{k+1}-x)+\ldots+P_n(a_n-x)$$

Carrying over the terms containing x to the left side of this equality and the rest to the right side, we obtain

$$(P_1+\ldots+P_k+P_{k+1}+\ldots+P_n)\,x=$$
$$=P_1a_1+\ldots+P_ka_k+P_{k+1}a_{k+1}+\ldots+P_na_n$$

Hence,

$$x=\frac{P_1a_1+\ldots+P_na_n}{P_1+\ldots+P_n}$$

Example 1. Let $P_1 = 1$, $P_2 = 2$, . . ., $P_n = n$; $a_1 = 1$, $a_2 = 2$, . . ., $a_n = n$. In such a case, from the developed

[1]) The term *abscissa of a point on the rod* from now on will be referred to as the distance from that point to the left end of the rod.

formula,

$$x = \frac{1^2 + 2^2 + \ldots + n^2}{1 + 2 + \ldots + n_l}$$

In order to calculate this relationship we will write the equalities

$$1^3 = (1 + 0)^3 = 1^3$$
$$2^3 = (1 + 1)^3 = 1^3 + 3 \cdot 1^2 + 3 \cdot 1 + 1$$
$$3^3 = (2 + 1)^3 = 2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(n + 1)^3 = (n + 1)^3 = n^3 + 3 \cdot n^2 + 3 \cdot n + 1$$

Adding these equalities we obtain

$$1^3 + 2^3 + \ldots + (n + 1)^3 = 1 + (1^3 + 2^3 + \ldots + n) + \\ + 3(1^2 + 2^2 + \ldots + n^2) + 3(1 + 2 + \ldots + n) + n$$

or

$$(n + 1)^3 = 1 + 3(1^2 + 2^2 + \ldots + n^2) + \\ + 3(1 + 2 + \ldots + n) + n$$

or

$$3(1^2 + 2^2 + \ldots + n^2) = (n + 1)(n^2 + 2n) - \\ - 3(1 + 2 + \ldots + n)$$

Dividing both parts by $3(1 + 2 + \ldots + n)$, we obtain

$$\frac{1^2 + 2^2 + \ldots + n^2}{1 + 2 + \ldots + n} = \frac{(n + 1)(n^2 + 2n)}{3(1 + 2 + \ldots + n)} - 1$$

The sum $1 + 2 + \ldots + n$ contains as many units as there are hatched squares in Fig. 31. But the number of these hatched squares equals that of unhatched squares, i.e. half of the total number of squares in the rectangle, $\frac{n(n+1)}{2}$. Therefore,

$$x = \frac{1^2 + 2^2 + \ldots + n^2}{1 + 2 + \ldots + n} = \frac{(n + 1)(n^2 + 2n)}{3(1 + 2 + \ldots + n)} - 1 =$$

$$= \frac{(n + 1)(n^2 + 2n)}{3\frac{n(n + 1)}{2}} - 1 = \frac{2n + 1}{3}$$

Hence, among other things, we have

$$1^2+2^2+\ldots+n^2=\frac{2n+1}{3}(1+2+\ldots+n)=$$

$$=\frac{2n+1n}{3}\frac{(n+1)}{2}=\frac{n(n+1)(2n+1)}{6}$$

Example 2. Let the same weights as in Example 1, i.e. weights of magnitudes 1, 2, . . ., n, be located at points

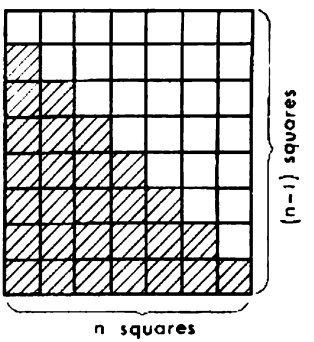

Fig. 31

with abscissas 1^2, 2^2, . . ., n^2. Prove that the abscissa of the center of gravity is

$$\frac{n(n+1)}{2}$$

(*Note*: From the results of Example 1 calculate the sum of cubes $1^3+2^3+\ldots+n^3$).

Example 3. Let the same weights as in Example 1, i.e. weights of magnitudes 1^2, 2^2, . . ., n^2 be located at points with abscissas 1, 2, . . ., n. Prove that the abscissa of the center of gravity is

$$\frac{3n(n+1)}{2(2n+1)}$$

Section 9. One problem from the theory of numbers (formulation)

Mechanical considerations prove on occasions to be helpful not only in geometrical problems but also in arithmetical ones. One rather unexpected example of this kind will now be considered.

We take a row of positive numbers $P_1, P_2, \ldots, P_n$; in order to shorten the row we subtract from the outer members P_1 and P_n the number P that is the smaller of the two. After that we add P to the middle members (if we have one middle member, then both numbers are added to this single middle number). In that case (if only $n > 2$) one or both outer members vanish; therefore the resulting row will be shorter than the original one. So if the original row was, for example,

1, 9, 3, 4

then we have the row

10, 4, 3

If the original row appeared as

6, 2, 6

then we obtain a row consisting of one number, namely,

14

The shorter row will be referred to as a "derivative" from the initial one. Thus, the row 10, 4, 3 is the derivative from the row 1, 9, 3, 4, and the row 14 is the derivative of the row 6, 2, 6. Note that for a row consisting of two numbers its derivative coincides with its own self.

Now we take any row α and proceed as follows. Let us find a derivative row from this row and denote it by α'. Next, we find a derivative row from α'. We denote it by α'' and call it the second derivative from the row α. The derivative from α'' will be referred to as the third derivative from α and designated by α'''. In general a derivative from a $(n-1)$th derivative will be referred to as the nth derivative and denoted by $\alpha^{\overbrace{n\text{ primes}}^{n'''\ldots'}}$. Since the length of every next derivative is by one or two units shorter than that of the preceding one, by calculating in succession the derivatives from the row α, we arrive at last at a row with length 1 or 2; this row will be called the last derivative, or characteristic, of the row α. So the characteristic of row 1, 9, 3, 4 is the row 7, 10.

Note that in passing from a row to its derivative the sum of its members does not change. The sum of all the members of the characteristic, therefore, equals the sum of all tho members of the original row; in the case when the characteristic consists of one member its value is equal to that sum. As with the case of a characteristic of length 2 we shall

learn to calculate the values of its members (without calculating the intermediate derivatives) in the section below. Now we pose the question of when the characteristic of a row includes one and when two members. First, let us try to clear up the question by way of a "mathematical experiment". To this end, calculate characteristics of several rows of the type 1, 2, 3, . . ., n. The consequent derivatives will be written one under another and the middle members will be underlined.

Example 1.

α 1 2 3
α' $\overline{4}$ 2

Example 2.

α 1 $\overline{2\ 3}$ 4
α' $\overline{3\ 4}$ 3
α'' $\overline{10}$

Example 3.

α 1 2 3 4 5
α' 2 $\overline{5}$ 4 4
α'' $\overline{7\ 6}$ 2
α''' 5 $\overline{10}$

Example 4.

α 1 2 3 $\overline{4\ 5}$ 5 6
α' 2 $\overline{4\ 5}$ 5 5
α'' 4 $\overline{9}$ 5 3
α''' 1 $\overline{12\ 8}$
α'''' $\overline{14}$ 7

Example 5.

α 1 2 3 4 5 6 7
α' 2 3 $\overline{6}$ 5 6 6
α'' 3 $\overline{8\ 7}$ 6 4
α''' 8 $\overline{13}$ 6 1
α'''' $\overline{7\ 14}$ 7
α''''' $\overline{28}$

Example 6.

α 1 2 3 4 5 6 7 8
α' 2 3 $\overline{5\ 6}$ 6 7 7
α'' 3 5 $\overline{10}$ 6 7 5
α''' 5 $\overline{13\ 9}$ 7 2
α'''' 3 13 $\overline{13}$ 7
α''''' $\overline{16\ 16}$ 4
α'''''' 12 $\overline{24}$

Example 7.

α	1	2	3	4	5	6	7	8	9
α'		2	3	4	$\overline{7}$	6	7	8	8
α''			3	4	$\overline{9\ 8}$		7	8	6
α'''				4	9	$\overline{14}$	7	8	3
α''''					1	9	$\overline{17}$	10	8
α'''''						9	$\overline{19}$	10	7
α''''''						2	$\overline{26}$	17	
α'''''''							$\overline{30}$	15	

The above examples indicate that the characteristic of a row constituting of a part of the natural series can involve both one and two members. We note the following regularity: a characteristic consists of one member if $n = 3k + 1$, otherwise, of two members. A new interesting feature manifests itself: if a characteristic consists of two members, then one of them is twice the other. Thus the following curious theorem suggests itself:

The characteristic of the row 1, 2, 3, . . ., n involves one member for $n = 3k + 1$ and two for $n = 3k$ or $n = 3k + 2$, in the latter case one of the two members (the first at $n = 3k$ and the second at $n = 3k + 2$) is twice the other.

This theorem will be proved in the following section. In so doing, the more general question shall not be overlooked: for what rows does the characteristic consist of one member, and for what of two. (How to determine that fact from numbers P_1, P_2, . . ., P_n without computating all the derivative rows?)

Section 10. One problem from the theory of numbers (solution)

Let us give our problem a new mechanical interpretation. Instead of a row of numbers P_1, P_2, . . ., P_n consider a row (Fig. 32*a*) loaded at points A_1, A_2, . . ., A_n (chosen so that $A_1A_2 = A_2A_3 = \ldots = A_{n-1}A_n$) with weights P_1, P_2, ..., P_n. Then a new system of weights will correspond to the derivative row.

We show that the material center of the derivative system of weights coincides with the material center of the initial system. Let us mentally split each of the extreme weights

P_1 and P_n into two weights

$$P_1 = P + (P_1 - P), \quad P_n = P + (P_n - P)$$

where P is the smaller of the weights P_1 and P_n (so that at any rate one of the weights $P_1 - P$ or $P_n - P$ is zero). Then the system of weights α might be represented as a union of two systems: a system α including weights $P_1 - P$, $P_2, \ldots, P_{n-1}, P_n - P$ (shown *under* the rod in Fig. 32*b*), and a system $\bar{\alpha}$ including two weights each of magnitude P (shown *over* the rod in Fig. 32*b*). In transition to the derivative system of weights the weights of system α remain in their places, but the weights of system $\bar{\alpha}$ come

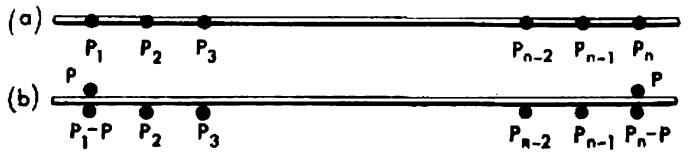

Fig. 32

closer together, each travelling an equal distance (in order to add to the middle members). In other words, the derivative system α' is a union of two systems $\underline{\alpha'}$ and $\bar{\alpha}'$, system $\underline{\alpha'}$ coinciding with system $\underline{\alpha}$ and system $\bar{\alpha}'$ resulting from system $\bar{\alpha}$ by bringing closer together both weights of $\bar{\alpha}$ to an equal distance. Since $\bar{\alpha}'$ and $\bar{\alpha}$ appear to be equivalent, the systems α and α', by virtue of Statement 3 in Sec. 5, are also equivalent. And the material centers of equivalent systems coincide.

So in transition from a system of weights to its derivative the material center of the system (hence the center of gravity too) does not change. Consequently, it does not change also in the transition from an original system to its characteristic.

Thus, the characteristic is one or two weights with the same center of gravity as the original system of weights. Moreover, it should be noted that locations of the point weights constituting the characteristic coincide with the locations of certain weights of the original system, i.e. with some of the points $A_1, A_2, \ldots, A_n$. If the characteristic contains two weights, those weights are to be found

at adjacent points A_j and A_{j+1}, with the center of gravity of the characteristic in between. Therefore, the characteristic will consist of one weight only when its center of gravity (or, which is just the same, the center of gravity of the original system) coincides with one of the points $A_1, \ldots, A_n$.

According to Sec. 8, the center of gravity of the initial system has the abscissa

$$\frac{P_1a_1+P_2a_2+\ldots+P_na_n}{P_1+P_2+\ldots+P_n}$$

where $a_1, \ldots, a_n$ are abscissas for the points $A_1, \ldots, A_n$. Putting $a_1 = 1$, $a_2 = 2, \ldots, a_n = n$, we can represent the abscissa of the center of gravity by the formula

$$\frac{P_1\cdot 1+P_2\cdot 2+\ldots+P_n\cdot n}{P_1+P_2+\ldots+P_n}$$

In order for the center of gravity to coincide with one of the integer points $A_1, A_2, \ldots, A_n$, it is necessary and sufficient that its abscissa be expressed by a whole number. Hence it finally follows that the *characteristic of the row* $P_1, P_2, \ldots, P_n$ *consists of one member if and only if the number* $\frac{P_1\cdot 1+P_2\cdot 2+\ldots+P_n\cdot n}{P_1+P_2+\ldots+P_n}$ *is an integer.*

In particular, if $P_1 = 1^2$, $P_2 = 2^2, \ldots, P_n = n^2$, then (see Example 3 in Sec. 8) the number $\frac{P_1\cdot 1+P_2\cdot 2+\ldots+P_n\cdot n}{P_1+\ldots+P_n}$ is equal to $\frac{3n(n+1)}{2(2n+1)}$. This number will never be an integer (except for the trivial case when $n = 1$). In fact, if that number were integer, then $4\,\frac{3n(n+1)}{2(2n+1)}$ would be an integer too. And in that case the difference

$$4\,\frac{3n(n+1)}{2(2n+1)}-(3n+1)=\frac{n-1}{2n+1}$$

would also be an integer. But the number $\frac{n-1}{2n+1}$ is an integer only when $n - 1 = 0$. Thus, with $n > 1$ the characteristic of the row $1, 4, 9, \ldots, n^2$ always consists of two numbers.

If $P_1 = 1$, $P_2 = 2, \ldots, P_n = n$, then (see Example 1 in Sec. 8) the magnitude of the expression $\frac{P_1\cdot 1+\ldots+P_n\cdot n}{P_1+\ldots+P_n}$

is the number $\frac{2n+1}{3}$. If $n = 3k + 1$, then the number $\frac{2n+1}{3} = 2k + 1$ is an integer and the characteristic consists of one member. If $n = 3k$, then $\frac{2n+1}{3} = 2k + \frac{1}{3}$; if $n = 3k + 2$, then $\frac{2n+1}{3} = (2k + 1) + \frac{2}{3}$; in the both last cases $\frac{2n+1}{3}$ is not an integer and the characteristic consists of two members. The above is the proof of the first part of the theorem formulated at the end of the preceding section.

If $n = 3k$, the center of gravity of the system of weights $1, 2, \ldots, n$ has, as shown above, an abscissa of $2k + \frac{1}{3}$ and thus is situated between the points A_{2k} and A_{2k+1}. It is, therefore, precisely at these points that the weights forming the characteristic of the original system are located. Since the center of gravity appeared to be twice as close to the point A_{2k} as to the point A_{2k+1}, of the two weights of the characteristic the one situated at A_{2k} is twice as big as the one located at A_{2k+1}. In such a way we obtain that when $n = 3k$, the first member of the characteristic is twice as large as the second one. If now $n = 3k + 2$, the abscissa of the center of gravity will be equal to $(2k + 1) + \frac{2}{3}$. Here the center of gravity is located between the points A_{2k+1} and A_{2k+2}, being twice as close to the second one of those points. The weight of the characteristic is, therefore, located at the second point, and is twice as big as the weight at the first point; in other words, the second member of the characteristic is twice as large as the first one. The theorem in Sec. 9 is thereby proved completely.

The sum of the members of the characteristic for the row $1, 2, \ldots, n$ equals the sum of the members of the row itself, i.e. $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$ (see Example 1 in Sec. 8). Knowing that one of the members is two times larger than the other, we may calculate those members as well. We obtain the following magnitudes for them: $\frac{n(n+1)}{6}$ (the smaller member) and $\frac{n(n+1)}{3}$ (the larger member).

Hence the characteristic of the row $1, 2, \ldots, n$ ap-

peares as

$$\frac{n(n+1)}{3}, \quad \frac{n(n+1)}{6} \quad (\text{if } n=3k)$$

$$\frac{n(n+1)}{2} \quad (\text{if } n=3k+1)$$

$$\frac{n(n+1)}{6}, \quad \frac{n(n+1)}{3} \quad (\text{if } n=3k+2)$$

To calculate the characteristic of the row $P_1, \ldots, P_n$ we can in general proceed as follows. First of all, find the number

$$x=\frac{P_1\cdot 1+P_2\cdot 2+\ldots+P_n\cdot n}{P_1+P_2+\ldots+P_n}$$

If x is an integer, the characteristic consists of one member equal to the sum $P_1 + P_2 + \ldots + P_n$. If x is not an integer, it must be represented as

$$x = y + z$$

where y is an integer and $0 < z < 1$. In that case, the characteristic consists of two numbers Q_1 and Q_2 such that

$$Q_1 + Q_2 = P_1 + P_2 + \ldots + P_n \qquad (10.1)$$

Since Q_1 and Q_2 can be thought of as weights located at points with abscissas y and $y + 1$ and having the center of gravity at a point with abscissa x, we have (according to Sec. 5)

$$\frac{Q_1}{Q_2}=\frac{1-z}{z} \qquad (10.2)$$

Solving (10.1) and (10.2) together we obtain Q_1 and Q_2

$$Q_1=\frac{1-z}{z}Q_2, \quad \frac{1}{z}Q_2=\frac{1-z}{z}Q_2+Q_2=Q_1+Q_2=P_1+\ldots+P_n$$

$$Q_2=(P_1+\ldots+P_n)z, \quad Q_1=(P_1+\ldots+P_n)(1-z) \qquad (10.3)$$

As the last example we find the characteristic of the row $1, 4, 9, \ldots, n^2$ $(n > 1)$. We have already seen that in this case $x = \frac{3n(n+1)}{2(2n+1)}$ and the characteristic involves two members. To obtain the members of the characteristic we must first find z. For this purpose put $n = 4k + r$, where $r = 0, 1, 2, 3$. Then

$$x=\frac{3n(n+1)}{2(2n+1)}=\frac{3(4k+r)(4k+r+1)}{2[2(4k+r)+1]}=3k+\frac{3}{2}\frac{2k+4kr+r+r^2}{8k+2r+1}$$

If $r = 0$, then $x = 3k + \frac{3}{2}\frac{2k}{8k+1} = 3k + \frac{3k}{8k+1}$; therefore, $z = \frac{3k}{8k+1} = \frac{v}{2n+1}$, where $v = 3k$.

If $r = 1$, then $x = 3k + \frac{3}{2}\frac{6k+2}{8k+3} = 3k + 1 + \frac{k}{8k+3}$; therefore, $z = \frac{k}{8k+3} = \frac{v}{2n+1}$, where $v = k$.

If $r = 2$, then $x = 3k + \frac{3}{2}\frac{10k+6}{8k+5} = 3k + 1 + \frac{7k+4}{8k+5}$; therefore, $z = \frac{7k+4}{8k+5} = \frac{v}{2n+1}$, where $v = 7k+4$.

If $r = 3$, then $x = 3k + \frac{3}{2}\frac{14k+12}{8k+7} = 3k + 2 + \frac{5k+4}{8k+7}$; therefore, $z = \frac{5k+4}{8k+7} = \frac{v}{2n+1}$, where $v = 5k + 4$.

The sum of the members of the characteristic is $1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$ (see Example 1 in Sec. 8). The members of the characteristic are obtained from the formula (10.3).

$$Q_1 = \frac{n(n+1)(2n+1)}{6}(1-z) = \frac{n(n+1)(2n+1)}{6}\frac{2n+1-v}{2n+1} = \frac{n(n+1)(2n+1-v)}{6}$$

$$Q_2 = \frac{n(n+1)(2n+1)}{6}z = \frac{n(n+1)(2n+1)}{6}\frac{v}{2n+1} = \frac{n(n+1)v}{6}$$

Substituting $4k$, $4k+1$, $4k+2$, $4k+3$ for n and taking into account the dependence of v from k, we can find the characteristic of the row $1, 4, 9, \ldots, n^2$:

$$\frac{2k(4k+1)(5k+1)}{3}, \quad 2k^2(4k+1) \quad (\text{if } n = 4k)$$

$$\frac{(2k+1)(4k+1)(7k+3)}{3}, \quad \frac{k(2k+1)(4k+1)}{3} \quad (\text{if } n = 4k+1)$$

$$\frac{(k+1)(2k+1)(4k+3)}{3}, \quad \frac{(2k+1)(4k+3)(7k+4)}{3} \quad (\text{if } n = 4k+2)$$

$$2(k+1)^2(4k+3), \quad \frac{2(k+1)(4k+3)(5k+4)}{3} \quad (\text{if } n = 4k+3)$$

Section 11. The perpetual motion machine is impossible

Let us consider a convex polygon with a point O inside it. Drop from point O perpendiculars to the sides of the

polygon. The feet of the perpendiculars can lie both on the sides proper (as, for example, the perpendicular to AB in Fig. 33), and on their extensions (as perpendiculars to the sides FA and EF in the same figure). Of course, there might be a case when the feet of all the perpendiculars coincide with the sides (that will be the case if, for instance, O is the center of a regular polygon). The question is: can it happen that the feet of all the perpendiculars coincide not with the sides but with their extensions? It appears

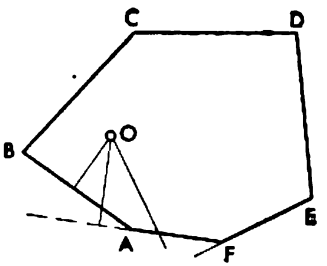

Fig. 33

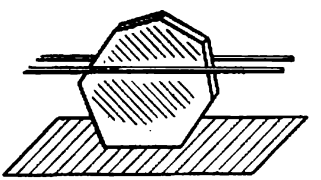

Fig. 34

that this cannot happen, i.e. the *foot of, at least, one perpendicular lies on the appropriate side* (*and not in its extension*). To prove that use will be made of the principle formulated in the name of this section. Suppose our polygon as a thin plate with the center of gravity at point O (e.g. it may be considered that the plate is weightless and at point O a point weight is located). Put the polygon with its side on a horizontal floor. If we position it strictly vertical, then it will not fall; if necessary it may be considered that it is supported at both sides by bars (Fig. 34). The polygon will either stand still or roll; if it rolls, it is bound to stop in the long run (otherwise we would have a perpetual motion machine). In the position in which it has stopped the perpendicular from the center of gravity to the support side must intersect this side (by virtue of Statement 2 in Sec. 5). Thus we have found the side of the polygon for which the foot of the perpendicular from point O lies on the side as such.

An analogous theorem holds for the polyhedron. Namely, let a convex polyhedron be given and a point O within it. Drop a perpendicular from that point to the planes of all the faces of the polyhedron. Then the *foot, at least, of one*

of these perpendiculars will lie on the face proper (*but not on its extension*).

To prove the above suppose a polyhedron as a mass with the center of gravity at O. (Let us, for example, consider the whole of the polyhedron weightless, and a point weight will be positioned at point O.) We put our polyhedron by one of its faces on a horizontal floor. The polyhedron will either stand still in equilibrium or roll; in the latter case it will, in the long run, stop in equilibrium. So the polyhedron has a position in which it stands still on the floor. Consider the face that is its support surface. By virtue of Statement 2 in Sec. 5 the perpendicular dropped from O on the surface of that face will traverse the face itself. Thus, we have found the face satisfying the requirements given in the theorem.

CONCLUSION

All the problems treated above could well be solved purely mathematically. One should not understand, however, that the employment of mechanical considerations in the solution of a mathematical problem is merely a mental exercise. Such methods are of historic and practical importance.

Even Archimedes used equilibrium considerations in finding the areas of curvilinear figures; so in calculating the surface area of a parabolic segment (i.e. a figure confined by a parabolic arc and a chord subtending the latter) he "suspended" the segment from an arm of a lever[1]). In all times mechanical and physical concepts had and still have a marked effect on production of mathematical results (if not as direct proofs, then as "leading" reasoning)[2]).

The applying of the laws of mechanics to mathematics is just a particular case of the general procedure consisting in the reverse use of connection of phenomena of nature with their mathematical description. To describe mathematically a phenomenon means to find formulae permitting to calculate physical characteristics of the phenomenon (speed, temperature, distance, etc.) or equations whose solutions serve to obtain the above-mentioned characteristics. The direct use of the mathematical description consists in that we can obtain numerical values of characteristics without observation of the phenomenon itself but performing calculations by appropriate formulae or solving appropriate equations. It is advantageous to proceed in such a way when the phenomenon in itself is complicated but its mathematical description is simple. One might, however, do the converse: instead of solving an equation one might carry out the phenomenon experimentally, measure desired characteristics and thereby obtain numerical results. Such a way is expedient when we deal with complex formulae or equations whose solution presents difficulties describing the phenomena susceptible to a relatively simple realization.

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[1]) See the following books: Kagan, V.F. *Archimedes.* Moscow Leningrad, Gostekhizdat, 1949. Lurie, S. Ya. *Archimedes.* Moscow-Leningrad, the Publishing House of the Acad. Sci. of the USSR, 1945.

[2]) See Ch. IX, Physical Mathematics, in *Mathematics and Plausible Reasoning* by G. Polya, Princeton, N.J. (1954).

This principle is the basis for modelling (i.e. investigation of physical processes from models), which follows the scheme: the investigation of process A (which is to be modelled); the description of the latter by formulae and equations E; at the same time E serves as a description for the process A' (the modelling one) carried out under laboratory conditions; the realization of process A' and the estimation by these means of the solution of equation E and, hence, the determination of required properties of process A. Process A (and also the formulae and equations E) are said to *be modelled* by the process A'. In addition, if two processes are described by the same formulae or equations, it is said that an *analogy* occurs between the two processes.

The solution of Steinhaus problem described in Sec. 4 is an elementary example of modelling. The phenomenon A being modelled is the going of children to school, and the characteristic of interest is the required time. This time is a value of a certain formula. A modelling phenomenon A' is found (suspension of weights from strings) for which the value of the same formula is that of the potential energy. The latter phenomenon is accomplished experimentally.

The modelling approach is of great practical importance[1]) (for the modelling use is made primarily of the hydrodynamical and electrical processes). In particular, this method forms the foundation for many types of so-called integrating machines; the principle of operation of the simplest of these, the friction wheel integrator, is described below.

Consider a straight line p and a curve l that is located at one side of the straight line (Fig. 35) so that each perpendicular to p intersects l at a single point. We will take two such perpendiculars AB and CD and will find the surface area S of the curvilinear trapezium $ABCD$. To this end we will break up the segment into minor parts by points

$$X_1 = A, \ X_2, \quad X_3, \ \ldots, X_n, \quad X_{n+1} = C$$

(in Fig. 36 $n = 6$). In each of the segments $X_i X_{i+1}$ select a point T_i from which a perpendicular is erected up to the intersection with curve l at a point Y_i. From each point Y_i draw straight lines parallel to p up to the perpendiculars

[1]) See items by Kholmogorov, A. N. "Simulation" and Gautenmacher, L. I. "Simulation in Mathematics". In: *The Great Soviet Encyclopaedia*, sec. ed., vol. 28.

erected from points X_i. The surface area S' of the hatched stepped figure in Fig. 36 is approximately equal to the surface area S of the curvilinear trapezium, the approximation

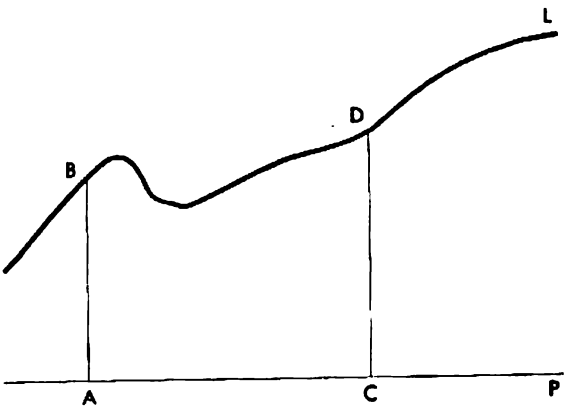

Fig. 35

being the more close, the smaller are the segments X_iX_{i+1} into which the base AC is divided. Let the lengths of the segments X_1, X_2, . . ., X_nX_{n+1} be equal to α_1, . . ., α_n,

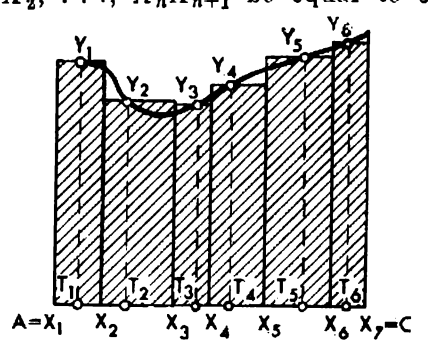

Fig. 36

respectively, and the lengths of the segments T_1Y_1, ... , T_nY_n be equal to y_1, . . ., y_n. Then the surface area S' will be

$$\alpha_1 y_1 + \ldots + \alpha_n y_n \qquad \text{(C.1)}$$

Now consider a system consisting of a disk *I* and a wheel *II* the rim of which is pressed to the disk *I* (Fig. 37). The wheel *II* can rotate about its axis. Let the distance from wheel *II* to the center *O* of the disk *I* be equal to y at any instant of time. Turn disk *I* through an angle α (in radians). The point on disk *I* that prior to the turn contacted wheel *II* will travel through an arc αy; exactly the same distance will be traversed by the point on the rim of wheel *II*. As that takes place wheel *II* will turn through an angle β.

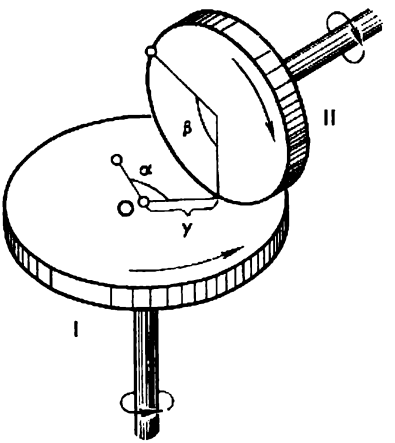

Fig. 37

Since in turning the wheel *II* through an angle β a point on its circumference describes an arc βr where r stands for the radius of wheel *II*, we have $\alpha y = \beta r$. Let us put $r = 1$, then $\beta = \alpha y$. If now we install the wheel *II* at distance y_1 from the center *O* and turn the disk through an angle α_1, next install wheel *II* at a distance y_2 from the center *O* and turn the disk through an angle α_2, etc., lastly, install wheel *II* at a distance y_n from the center *O* and turn the disk through an angle α_n, then, as a result, the total angle through which the wheel will be turned will be numerically equal to the value (C.1), i.e. the surface area S' of the hatched stepped figure in Fig. 36.

Draw the straight line p and the curve l on a sheet of paper (Fig. 38). Consider the following mechanism. A rod is installed on two small wheels one of which rolls along

the straight line p (as the rod is always kept normal to this straight line). A runner fitted onto the rod is capable of moving along the latter. In the center of the runner a pin is provided with the end touching the paper. Suppose the rod is connected with the disk I and the wheel II in the following way: a turn of the small wheel of the rod results in a turn of disk I through the same angle; the distance from wheel II to the center O of the disk at any instant equals that of the center of the runner to the beginning of the rod. Assume the radius of the rod equals unity. Contour the stepped figure in Fig. 38 with the pin, the rod rolling along

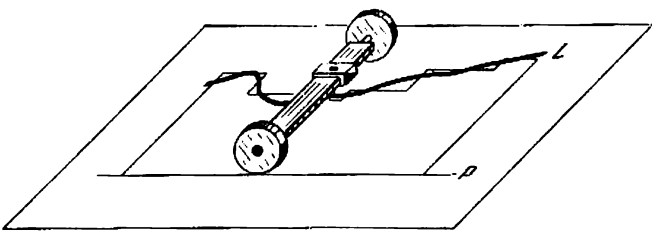

Fig. 38

the straight line p and the runner moving along the rod. It follows as a result, that the angle through which wheel II turns is numerically equal to the surface S'. The smaller are these elementary parts of segment AC, the closer is the approximation of curve l by the broken line, which constitutes the contour of the stepped figure, and hence of S by S'. If, therefore, we contour not the broken line but the curve l itself, wheel II will turn through angle equal to S. The result may be recorded on a special scale.

The modelling process here is that of measuring the surface area of the curvilinear trapezium through the calculation of the surface areas of stepped figures. This process is described by formula (C.1). The same formula appeared to describe some other process associated with the friction mechanism shown in Fig. 37. This offers the opportunity of modelling, through the shown mechanism formula (C.1) and the procedure of measurement of the surface area.

The mechanical analogies have led us in such a manner

to a device for calculating the surface areas of curvilinear figures (true, of special figures—curvilinear trapezia)[1]).

In conclusion, we will make one observation of a principal character. The reasoning based on mechanical considerations might appear to the fastidious reader not rigorous enough. Even the first simplest example treated in Sec. 1 is apt to give rise to objections. The string actually is not a straight line (it can have finite thickness and if examined through a microscope will have a rather irregular form), so the phrase "the string is a straight line" has no correct meaning. Next, strictly speaking, the end of the string cannot be referred to as a point (besides, the very statement that a weight is attached to a string at a given point again is not faultless). At last, any actual string is extensible, and therefore the length of a string with a weight is larger than the length of the string without the weight.

One might well make many critical remarks of this kind, pertaining both to the above reasoning and like considerations underlying all the treatment. One should not believe, however, that these considerations are, thereby, deprived of their instructive value. We do consider them no less convincing than the geometrical proofs. The point is that all our propositions of strings, weights and so on, are of an approximate character only and are the more valid the more ideal are the properties of a chosen object (string weight, etc.). In our case certain statements (e.g. "the string is a straight line") hold the more correct the thinner is a selected string; and it is said in such a case that these propositions are valid for infinitely thin string. Other provisions (e.g. that the length of a string with a weight and without a weight is the same) hold the more correct the less extensible is the string; in such a case it is said that these propositions are true for an absolutely unextensible string. It is clear that infinitely thin and absolutely unextensible strings don't exist in nature. This is as an idealized concept, as, for example, that of the point of application of a force (known from the elementary physics); the conclusions made for forces applied to points are satisfied actually the more closely the less is the surface of interac-

[1]) Devices indicating the surface area of a curvilinear trapezium in contouring its curved side are called integrimeters, and those indicating the surface area of any figure in complete contouring of the latter are called planimeters.

tion of bodies. The same sort of idealization is the concept of a particle; the properties obtained for particles hold for real bodies the more true the smaller their volumes.

It is well to emphasize here that all the concepts and laws of mechanics (like those of physics in general) are associated with the idealization of natural phenomena; therefore even a student, may be unconsciously, not infrequently encounters such idealizations. Take, for example, one of the basic concepts of mechanics, that of uniform motion. Absolutely uniform motion is absent in nature (careful measurement indicates that even the arms of a clock move nonuniformly). In a number of cases this nonuniformity is so small that it may be neglected; it is in such a way that the concept of uniform motion developed. The same could be said about the first law of Newton, because there are no bodies unaffected by other bodies. The list of such examples could be continued endlessly, they go through all physics.

Thus we see that mechanics deals, in fact, not with real but idealized bodies and processes, such as "a body unaffected by other bodies", "uniform linear motion", and so on. In this case, we treat these idealized objects as real, i.e. as if they occurred in reality.

Note, at last, that mechanical abstractions (such as a particle, a weightless, infinitely thin, and absolutely unextensible string and so on) in their nature are as good as geometrical ones (such as a point, straight line, plane and so on). Points, straight lines and planes do not occur in real life as real objects do, and the true meaning of statements involving them consists in that they are the more valid the closer are the real objects to points, straight lines, planes, etc., respectively, in their properties.

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